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Abstract

Full Text

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SOME PROPERTIES OF UNCONDITIONAL BASES

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In the present note it is shown that in some cases the topology of a vector space is uniquely (up to isomorphism) determined solely by specifying the class of all numerical sequences $\{a_k\}_1^\infty$ for which the series $\sum_{k=1}^\infty a_k x_k$ converge in this topology (here $\{x_k\}_1^\infty$ is a fixed sequence of elements of the space). The characteristic result in this question is Theorem 3a.

In what follows we consider Banach spaces; $E\{x_k\}_1^\infty$ denotes the closed linear span of the sequence $\{x_k\}_1^\infty$, and $\{x_{k_n}\}_{n+1}^\infty$ is the subsequence of $\{x_n\}_{n=1}^\infty$ consisting of those and only those elements which did not enter the subsequence $\{x_{k_n}\}_{n=1}^\infty$, i.e.

$$\{x_{\bar{k}_n}\}_1^\infty = \{x_n\}_1^\infty \setminus \{x_{k_n}\}_{n=1}^\infty.$$

We shall say that the sequences $\{x_n\}_1^\infty \subset B_1$ and $\{y_n\}_1^\infty \subset B_2$ (B_1 and B_2 are Banach spaces) are **isomorphic** if there exists a bounded and invertible operator A carrying $E\{x_n\}_1^\infty$ onto $E\{y_n\}_1^\infty$ such that $\{Ax_n = y_n\}_1^\infty$. The sequence $\{x_n\}_1^\infty$ is called **minimal** if there exists a biorthogonal sequence $\{f_n\}_{n=1}^\infty$ ($f_n(x_k) = \delta_{nk}$) of linear functionals.

Lemma 1. *Let the sequence $X = \{x_n\}_1^\infty$, together with any subsequence obtained from X by deleting a finite number of terms, not be minimal. Then from the sequence X one can delete an infinite number of elements in such a way that the closed linear span of the remaining ones coincides with $E\{x_n\}_1^\infty$.*

In what follows we shall use the notion of the **inclination**$*$ $\delta(B_1, B_2)$ of two subspaces $B_1, B_2 \subset B$:

$$\delta(B_1, B_2) = \inf_{\substack{x\in B_1,\ y\in B_2, \\ \|x\|=1}} \|x + y\|.$$

Theorem 1. *Let $\{x_n\}_1^\infty \subset B$ and $\delta(E\{x_{k_n}\}_{n=1}^\infty, E\{x_{\bar{k}_n}\}_{n=1}^\infty) > 0$** for every infinite subsequence $\{k_n\}_{n=1}^\infty$.*

Then $\{x_n\}_1^\infty$ is an unconditional basis in $E\{x_n\}_1^\infty$.

Theorem 2. *Let $\{x_n\}_1^\infty$, $\|x_n\| \neq 0$, be a sequence of elements of a Banach space B, and let every convergent series $\sum_{k=1}^\infty a_k x_k$ converge unconditionally. Then* *either a) from the sequence $\{x_k\}_1^\infty$ one can delete a finite number of elements in such a way that the remaining sequence*

$*$ The term inclination we borrowed from ([1]). This notion is, in a certain sense, equivalent to the notion of minimal angle from ([2], [3]).

$**$ We do not require boundedness away from zero here; in this the criterion formulated differs from the known one.

is an unconditional basis in its closed linear span, or b) there exists a numerical sequence $\{a_k\}_1^\infty$ such that the sequence

$$\left\{u_n = \sum_{N_n+1}^{N_{n+1}} a_k x_k\right\}_{n=1}^\infty$$

is isomorphic to the natural basis of the space c_0.

Proof. Suppose that there exists an infinite subsequence $\{x_{k_n}\}_{n=1}^\infty$ such that

$$\delta(E\{x_{k_n}\}_{n=j}^\infty,\, E\{x_{\bar{k}_n}\}_{n=j}^\infty) = 0 \tag{1}$$

for every $j = 1, 2, \ldots$.

Denote $E\{x_{k_n}\}_1^\infty = B_1,\ E\{x_{\bar{k}_n}\}_1^\infty = B_2$. Equality (1) means that for any sequence $\{\varepsilon_n\}_1^\infty,\ \varepsilon_n > 0$, there exists a sequence of elements $\{y_n\}_1^\infty,\ \|y_n\| = 1$, possessing the properties:

$$y_{2n-1} \in B_1, \qquad y_{2n} \in B_2;$$

$$y_{2n-1} = \sum_{k=m_0+1}^{m_{n+1}}{}' d_k x_k, \qquad y_{2n} = \sum_{k=m_n+1}^{m_{n+1}}{}'' d_k x_k, \qquad \|y_{2n} - y_{2n-1}\| < \varepsilon_n.$$

Let

$$\sum_{n=1}^\infty \varepsilon_n < \infty. \tag{2}$$

Denote

$$c_{2n-1} = \sup_{k \geq m_n+1} \left\| \sum_{j=k}^{m_{n+1}}{}' d_{jx} j \right\| = \left\| \sum_{k_{2n-1}^0}^{m_{n+1}}{}' d_{jx} j \right\|,$$

$$c_{2n} = \sup_{k \geq m_n + 1} \left\| \sum_{j=k}^{m_{n+1}} {}'' d_{jx} j \right\| = \left\| \sum_{k_{2n}^0}^{m_{n+1}} {}'' d_{jx} j \right\|;$$

$$z_{2n-1} = \frac{1}{c_{2n-1}} \sum_{k_{2n-1}^0}^{m_{n+1}} {}' d_{jx} j, \qquad z_{2n} = \frac{1}{c_{2n}} \sum_{k_{2n}^0}^{m_{n+1}} {}'' d_{jx} j; \qquad \max(c_{2n}, c_{2n-1}) = c_n^* \geq 1.$$

For any sequence of numbers $b_k \xrightarrow[k \to \infty]{} 0$, the series

$$\sum_{k=1}^{\infty} b_k \frac{y_{2k} - y_{2k-1}}{c_k^*} = \sum_{j=1}^{\infty} \alpha_{jx} j \tag{3}$$

converges (by virtue of (2) and the special choice of c_n^*) and, therefore, by the hypothesis of the theorem, converges unconditionally. Choose an infinite sequence of indices $\{k\}$ (we shall not introduce new notation for it) such that $c_{2k}^* = c_{2k}$ (or $c_k^* = c_{2k-1}$). The unconditional convergence of the series (3) means that the series

$$\sum_{k=1}^{\infty} b_k \frac{y_{2k}}{c_{2k}} \quad \text{and the series} \quad \sum_{k=1}^{\infty} b_k z_{2k} \quad (\|z_{2k}\| = 1)$$

converge unconditionally for every sequence $b_k \to 0$ $(k \to \infty)$. The sequence $\{z_{2k}\}_{k=1}^{\infty}$ converges weakly to zero (otherwise there would exist a sequence $\{b_k\}_{k=1}^{\infty}$, $b_k \xrightarrow[k \to \infty]{} 0$, for which the series

$$\sum_{k=1}^{\infty} b_k z_{2k}$$

would diverge). By Pelczyński' s theorem ([4]), from the sequence $\{z_{2k}\}_1^{\infty}$ one can extract a basic sequence $\{u_n\}_1^{\infty}$. Moreover, for

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* Σ' means that the summation is over $\{k_n\}_{n=1}^{\infty}$, and Σ'' over $\{\bar{k}_n\}_{n=1}^{\infty}$.

for every sequence $\{b_k\}_1^{\infty}$, $b_k \to 0$, there exists an element of the space

$$x = \sum_{k=1}^{\infty} b_k u_k.$$

Moreover, since $\{u_n\}_1^{\infty}$ is a basis in $E\{u_n\}_1^{\infty}$, there exists a constant K such that

$$K \max_k |b_k| \leq \left\| \sum_{k=1}^{\infty} b_k u_k \right\|.$$

By the Banach theorem on the inverse operator, the sequence $\{u_n\}_1^\infty$ is isomorphic to the natural basis of the space c_0 (in particular, $E\{u_n\}_1^\infty \cong c_0{}^*$).

Suppose now that there is no infinite subsequence $\{x_{k_n}\}_{n=1}^\infty$ for which (1) holds. Then, by Lemma 1, from the sequence $\{x_n\}_1^\infty$ one can remove a finite number of elements so that the remaining sequence is minimal. Applying Theorem 1 to the minimal sequence, we obtain that it is an unconditional basis. This proves the theorem.

Remark. It is easy to give an example showing that part b) of the theorem is essential.

Theorem 3a. *Let $\{x_k\}_1^\infty \subset B$, and let the series*

$$\sum_{k=1}^\infty a_k x_k$$

converge if and only if

$$\sum_{k=1}^\infty |a_k|^p < \infty \quad (1 \le p < \infty).$$

Then $E\{x_k\}_1^\infty \cong l_p$; moreover, from the sequence $\{x_k\}_1^\infty$ one can remove a finite number of elements so that the remaining sequence is isomorphic to the natural basis of l_p.

Proof. From the convergence of the series

$$\sum_{k=1}^\infty a_k x_k$$

there follows its unconditional convergence, since convergence is determined only by the behavior of the moduli of the coefficients. Thus, Theorem 2 is applicable to the sequence $\{x_k\}_1^\infty$. We shall show that part b) does not occur. Otherwise there exists the sequence specified in the theorem

$$u_n = \sum_{k=m_n+1}^{m_{n+1}} a_k x_k, \quad \|u_n\| = 1.$$

Denote

$$\sum_{k=m_n+1}^{m_{n+1}} |a_k|^p = d_n^p.$$

For any sequence $|b_n|^p \to 0$, the series

$$\sum_{n=1}^\infty b_n u_n$$

converges, and hence

$$\sum_{n=1}^{\infty} |b_n|^p d_n^p < \infty.$$

It follows that

$$\sum_{n=1}^{\infty} d_n^p < \infty.$$

But then the series

$$\sum_{k=1}^{\infty} u_k$$

must converge, which is impossible since $\|u_k\| = 1$. Thus, from the sequence $\{x_k\}_1^\infty$ one can remove a finite number of elements so that the remaining sequence is an unconditional basis in $E\{x_k\}_1^\infty$. The assertion of the theorem now follows by application of the following lemma.

Lemma 2. *Let $\{y_k\}_1^\infty \subset B_1$ and let it be an unconditional basis in B. Let the sequence $\{x_k\}_1^\infty \subset B_2$ have the following property: for every numerical sequence $\{a_k\}_1^\infty$ for which there exists*

$$y = \sum_{k=1}^{\infty} a_k y_k \in B_1,$$

the series

$$\sum_{k=1}^{\infty} a_k x_k$$

converges. Then there exists a constant C

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* $B_1 \cong B_2$ means that the space B_1 is isomorphic to the space B_2.

(not depending on $\{a_k\}_1^\infty$) such that

$$\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_2 \le C \left\| \sum_{k=1}^{\infty} a_k y_k \right\|_1 .$$

Theorem 3 formulated below contains Theorem 3a in a trivial way. We have singled out Theorem 3a because it was of special interest to us.

Theorem 3. Let $\{e_k\}_1^\infty$ be an unconditional basis in B_1, and suppose that B_1 contains no subspace isomorphic to c_0, while $\{x_k\}_1^\infty$ is some sequence of elements of B_2. Let the series $\sum_{k=1}^{\infty} a_k x_k$ converge in B_2 if and only if the series $\sum_{k=1}^{\infty} a_k e_k$ converges in B_1. Then the space $E\{x_k\}_1^\infty$ is isomorphic to some subspace of B_1 with finite defect; moreover, from the sequences $\{x_k\}_1^\infty$ and $\{e_k\}_1^\infty$ one can delete a finite number of elements (with the same indices) so that the remaining sequences are isomorphic.

We shall not give the proof of this theorem here, since it is constructed in the same way as the proof of Theorem 3a. One need only apply Lemma 3.

Lemma 3. In order that the basis $\{x_k\}_1^\infty$ be boundedly complete*, it is necessary and sufficient that, for every sequence $\{u_j\}_{j=1}^\infty$, $u_j \subset E\{x_k\}_{k=n_j+1}^{n_{j+1}}$, such that the series $\sum_{j=1}^\infty b_j u_j$ converges for every monotone sequence $b_j \to 0$, the series $\sum_{j=1}^\infty u_j$ converge.

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* For the definition and properties of a boundedly complete basis, see, for example, (5). Let us note only that for an unconditional basis $\{e_k\}_1^\infty$ the requirement of bounded completeness is equivalent to the absence in $E\{e_k\}_1^\infty$ of a subspace isomorphic to c_0.

Note: Figure translations are in progress. See original paper for figures.