HYDROMECHANICS

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1965

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Abstract

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NONLINEAR THEORY OF THE ONSET OF TURBULENCE IN FLOWS WITH A GRADIENT OF THE MEAN VELOCITY

(Presented by Academician A. N. Kolmogorov on 18 XI 1964)

I. Let us consider the development of small initial velocity pulsations in a flow of a viscous incompressible fluid. Let the velocity of the averaged flow be maintained unchanged in such a way that $\partial U_1/\partial x_2 = \text{const}$, $U_2 \equiv U_3 \equiv 0$. For the velocity pulsations we have the Navier–Stokes equations

$$\frac{\partial \delta V_i(\mathbf{x},t)}{\partial t} + U_j \frac{\partial}{\partial x_j}\delta V_i + \delta V_j \frac{\partial U_i}{\partial x_j} + \delta V_j \frac{\partial \delta V_i}{\partial x_j} = \nu \Delta \delta V_i - \frac{1}{\rho}\frac{\partial \delta P(\mathbf{x},t)}{\partial x_i}. \tag{1}$$

The energy $E(t)$ of vortex disturbances $\delta\mathbf{V}(\mathbf{x},t)$ with infinitesimally small initial amplitude A (and scale l of the initial vortices) in such a flow increases and reaches a certain maximum, and then decays; the quantity $\max_{0<t<\infty} E(t)/E(0)$ depends on the Reynolds number $\text{Re} = (\partial U_1/\partial x_2)(l^2/\nu)$ and increases without bound as Re increases ([1]). Let us consider the development of pulsations with a finite initial amplitude.

Introduce the Fourier transform of the pulsation velocity

$$\widetilde{\mathbf{V}}(\mathbf{k};t) = \int_{-\infty}^{\infty} e^{i\mathbf{k}\mathbf{x}} \delta\mathbf{V}(\mathbf{x},t)\, d\mathbf{x}. \tag{2}$$

From the Navier–Stokes equation, eliminating the pressure by means of the incompressibility condition, we obtain the following equation for $\widetilde{\mathbf{V}}(\mathbf{k};t)$ (in a flow with constant gradient $\partial U_1/\partial x_2$):

$$\frac{\partial \widetilde{V}_i(\mathbf{k};t)}{\partial t} - \frac{\partial U_1}{\partial x_2} k_1 \frac{\partial}{\partial k_2}\widetilde{V}_i(\mathbf{k};t) + \frac{\partial U_1}{\partial x_2}\delta_{i1}\widetilde{V}_2 + \nu k^2 \widetilde{V}_i -$$

$$-2\frac{\partial U_1}{\partial x_2}\frac{k_1 k_2}{k^2}\widetilde{V}_2 = i\left(\frac{k_i k_l}{k^2} - \delta_{il}\right)\int_{-\infty}^{\infty} k'_j \widetilde{V}_l(\mathbf{k}';t)\widetilde{V}_j(\mathbf{k}-\mathbf{k}';t)\, d\mathbf{k}'. \tag{3}$$

Let us consider a plane-parallel flow $\widetilde{\mathbf{V}} = (V_1(k_1;k_2;t);\ V_2(k_1;k_2;t))$. In this case, from the incompressibility condition $k_1\widetilde{V}_1 + k_2\widetilde{V}_2 = 0$, we find

$$\widetilde{V}_1 = -k_2 V(k_1;k_2;t), \qquad \widetilde{V}_2 = k_1 V(k_1;k_2;t). \tag{4}$$

Then from (3), (4) follows the nonlinear equation for the function $V(\mathbf{k};t)$:

$$\frac{\partial V(\mathbf{k};t)}{\partial t} - \frac{\partial U_1}{\partial x_2}k_1\frac{\partial}{\partial k_2}V(\mathbf{k};t) + \nu k^2 V(\mathbf{k};t) - 2\frac{\partial U_1}{\partial x_2}\frac{k_1k_2}{k^2}V(\mathbf{k};t) =$$

$$= \frac{i}{k^2}\int_{-\infty}^{\infty}(k_2k_1' - k_1k_2')(k_2k_2' + k_1k_1')V(\mathbf{k}';t)V(\mathbf{k}-\mathbf{k}';t)\,d\mathbf{k}'. \tag{5}$$

Let us consider a solution of equation (5) with the initial condition

$$V(\mathbf{k};0) = (A/\alpha)\exp(-k^2/\alpha^2),$$

which corresponds to vortex initial disturbances. Put $V(\mathbf{k};t) = D(\mathbf{k};t) + iB(\mathbf{k};t)$. Introduce the dimensionless Reynolds number $\mathrm{Re} = (\partial U_1/\partial x_2)(1/\alpha^2\nu)$ and pass to dimensionless variables $k^* = k/\alpha$, $t^* = t\,\partial U_1/\partial x_2$; $A^* = A\alpha/(\partial U_1/\partial x_2)$. From the parameters A, α, ν, $\partial U_1/\partial x_2$ one can form two independent dimensionless combinations: $\mathrm{Re} = (\partial U_1/\partial x_2)(1/\alpha^2\nu)$ and $A^* = A\alpha/(\partial U_1/\partial x_2)$. At a fixed value of A^*, the number Re may increase through a decrease of the viscosity ν. (The index $*$ on k^* and t^* will be omitted below.) We shall seek a solution of equation (5) by the method of successive approximations.

In the first, linear, approximation we have

$$D_1(\mathbf{k};t) = A^*\left(\frac{k_1^2 + (k_2 + k_1t)^2}{k_1^2 + k_2^2}\right)e^{-k_1^2-(k_2+k_1t)^2}\times$$

$$\times\exp\left\{-\frac{1}{\mathrm{Re}}\left(k^2t + k_1k_2t^2 + \frac{t^3k_1^2}{3}\right)\right\}; \qquad B_1(\mathbf{k};t) = 0. \tag{6}$$

Denote $D_1(\mathbf{k};t)/A^* = G(\mathbf{k};t)$. The solution is represented in the form of a series in powers of A^*:

$$V(\mathbf{k};t) = \sum_{n=1}^{\infty}\{D_n(\mathbf{k};t) + iB_n(\mathbf{k};t)\}, \tag{7}$$

where $D_n(\mathbf{k};t)$ and $B_n(\mathbf{k};t)$ are solutions of the linear nonhomogeneous equations following from equations (5), (6).

Analysis of these equations shows that $D_{2m}(\mathbf{k},t) = B_{2m-1}(\mathbf{k},t) = 0$, and for $D_{2m+1}(\mathbf{k};t)$ and $B_{2m}(\mathbf{k};t)$ the following estimates hold.

For $t \ll 1$:

$$|B_{2m}(\mathbf{k}, t)|,\ |D_{2m+1}(\mathbf{k}; t)| < \frac{(A^* K_1)^{2m+1}}{2m!}\, t^{2m} G(\mathbf{k}; t). \tag{8}$$

For $t > 2\,\mathrm{Re}^{1/3}$:

$$A^* \left(\frac{A^*\,\mathrm{Re}^{3/2}}{K_2 t^3}\right)^{2m} G_{2m+1}(\mathbf{k}; t) < |D_{2m+1}(\mathbf{k}; t)| < A^* \left(\frac{A^*\,\mathrm{Re}^2}{K_2}\right)^{2m} G_{2m+1}(\mathbf{k}; t); \tag{9}$$

$$A^* \left(\frac{A^*\,\mathrm{Re}^{3/2}}{K_2 t^3}\right)^{2m-1} G_{2m}(\mathbf{k}; t) < |B_{2m}(\mathbf{k}; t)| < A^* \left(\frac{A^*\,\mathrm{Re}^2}{K_2}\right)^{2m-1} G_{2m}(\mathbf{k}; t). \tag{10}$$

Here $K_1, K_2 = \mathrm{const}$;

$$G_{2m+1}(\mathbf{k}; t) = \left[\frac{k_1^2 + (k_2 + k_1 t)^2}{k_1^2 + k_2^2}\right] \exp\left\{-(k_1^2 + (k_2 + k_1 t)^2)\frac{1}{2m+1} - \frac{1}{\mathrm{Re}(2m+1)}\left(k^2 t + k_1 k_2 t^2 + \frac{t^3 k_1^2}{3}\right)\right\}.$$

The derivation of estimates (8)–(10) is based on the following relations. Consider the integral

$$|I(k_1; k_2; t)| = \frac{1}{k^2}\int_{-\infty}^{\infty} (k_2 k_1' - k_1 k_2')(k_2 k_2' + k_1 k_1') G(\mathbf{k}'; t)\, G(\mathbf{k} - \mathbf{k}'; t)\, d\mathbf{k}'. \tag{11}$$

For $t \to 0$ we have $I(k_1; k_2; t) = O(1)$.

For $t \to \infty$,

$$\exp\left\{-\frac{2(k_1' - k_1/2)^2 t^3}{3\,\mathrm{Re}}\right\} \Rightarrow \sqrt{\frac{3\,\mathrm{Re}}{2t^3}}\, \delta(k_1' - k_1/2).$$

For $t > 2\,\mathrm{Re}^{1/3}$ the integral over infinite limits in (11) can be "cut off," estimated by the integral over the region

$$|k_1' - k_1/2| \le 3\sqrt{\frac{3\,\mathrm{Re}}{2t^3}}; \qquad |k_2' - k_2/2| \le \min\left(3;\ 3\sqrt{\frac{\mathrm{Re}}{2t}}\right);$$

then we obtain the estimate (for $t > 2\,\mathrm{Re}^{1/3}$):

$$|I(k_1;k_2;t)| > \exp\left\{-\frac{k_1^2}{2} - \frac{(k_2+k_1t)^2}{2} - \frac{k^2t + k_1k_2t^2 + t^3k_1^2/3}{2}\right\}\times$$

$$\times\left(\frac{k_1^2+(k_2+k_1t)^2}{k_1^2+k_2^2}\right)^2 \frac{\mathrm{Re}^{1/2}}{16k^2t^2}\,|k_1k_2^3 + \text{terms of lower degree in } t, k_2|. \quad (12)$$

Using estimates analogous to (12), and the properties of the linear nonhomogeneous equations for $D_n(\mathbf{k};t)$, $B_n(\mathbf{k};t)$, by mathematical induction one can establish relations (8)–(10) for all $m \geq 1$.

From the obtained relations (7)–(10) it follows that, for $A^* < K_2/\,\mathrm{Re}^2$, the series (7) converges for all t and gives a solution $V(\mathbf{k};t)$ that decreases as $t \to \infty$. For $A^* > K/\,\mathrm{Re}^{1/2}$, the series (7) converges only for small t, becoming infinite at large t.

From this one may conclude that there exists a "stability barrier"

$$K_2/\,\mathrm{Re}^2 \leq \bar{A}^*(\mathrm{Re}) \leq K/\,\mathrm{Re}^{1/2} \quad (13)$$

for the initial amplitude A of vortex pulsations such that, if

$$A < (\partial U_1/\partial x_2)(1/a)\bar{A}^*(\mathrm{Re}),$$

then the pulsations decay as $t \to \infty$, whereas if

$$A > (\partial U_1/\partial x_2)(1/a)\bar{A}^*(\mathrm{Re}),$$

the vortex pulsations (in a free stream with a constant mean-velocity gradient) grow and their energy

does not decrease, and that as $\mathrm{Re} \to \infty$ the "stability barrier" $A^*(\mathrm{Re})$ contracts to zero.

II. Let us analyze the nonlinear mechanism of the maintenance of non-decaying turbulence on the basis of equations (3) for $\hat{V}(\mathbf{k};t)$, and also on the basis of the differential equations for the spectral tensors of turbulent pulsations ([1,2]), including higher moments.

We prescribe the initial conditions in the form

$$V(\mathbf{k};0) = A^0[\delta(\mathbf{k}-\mathbf{k}_0) + \delta(\mathbf{k}+\mathbf{k}_0)], \quad (14)$$

which corresponds to a plane wave with wave vector $\mathbf{k}_0$. Then all waves that are multiples of $\mathbf{k}_0$ will appear in the solution of equation (3). If

$$V(\mathbf{k};0) = A^0[\delta(\mathbf{k}-\mathbf{k}_1) + \delta(\mathbf{k}+\mathbf{k}_1) + \delta(\mathbf{k}-\mathbf{k}_2) + \delta(\mathbf{k}+\mathbf{k}_2)], \qquad (15)$$

which corresponds to two plane waves with wave vectors $\mathbf{k}_1, \mathbf{k}_2$, then all waves with wave vectors $n\mathbf{k}_1$; $m\mathbf{k}_2$; $n\mathbf{k}_1 - m\mathbf{k}_2$; $n, m = \pm 1, \pm 2, \pm 3, ...$ will arise. In this case, with the passage of time there will appear not only smaller-scale but also larger-scale pulsations (since $|n\mathbf{k}_1 - m\mathbf{k}_2| < |\mathbf{k}_1|, |\mathbf{k}_2|$ for certain n, m), which will grow longer and more strongly, drawing energy directly from the mean flow. If $\mathbf{k}_1, \mathbf{k}_2$ are incommensurable, then with time waves are excited whose wave vectors form an everywhere dense set. At the same time large-scale pulsations will continually appear, drawing energy from the gradient of the mean flow and transferring it to the small-scale pulsations arising from them.

Solutions of equations (3), as well as solutions of the closed system of equations for the moments, including fourth or fifth moments ([1,2]), have the indicated character. In the approximation with fourth moments the closed system is formed by equations (5), (6) from ([2]) and by the relations following from the hypothesis of a Gaussian relation between fourth and second moments (the notation here is the same as in ([1,2])):

$$\Phi_{il,j,k}(\mathbf{k}_1;\mathbf{k}_2) = \Phi_{lk}(\mathbf{k}_2)\Phi_{ij}(\mathbf{k}_1-\mathbf{k}_2) + \Phi_{ik}(\mathbf{k}_2)\Phi_{lj}(\mathbf{k}_1-\mathbf{k}_2);$$

$$\Phi_{i,lj,k}(\mathbf{k}_1;\mathbf{k}_2) = \Phi_{il}(\mathbf{k}_1)\Phi_{jk}(\mathbf{k}_2) + \Phi_{ij}(\mathbf{k}_1)\Phi_{lk}(\mathbf{k}_2); \qquad (16)$$

$$\Phi_{i,j,lk}(\mathbf{k}_1;\mathbf{k}_2) = \Phi_{il}(\mathbf{k}_1)\Phi_{jk}(\mathbf{k}_2-\mathbf{k}_1) + \Phi_{ik}(\mathbf{k}_1)\Phi_{jl}(\mathbf{k}_2-\mathbf{k}_1).$$

If at the initial instant one plane wave is excited, then the second-order spectral tensor $\Phi_{ij}(\mathbf{k};0)$ is equal to

$$\Phi_{ij}(\mathbf{k};0) = A^2(\delta_{ij}k^2 - k_ik_j)[\delta(\mathbf{k}-\mathbf{k}_0) + \delta(\mathbf{k}+\mathbf{k}_0)]. \qquad (17)$$

At the initial instant $\Phi_{ij}(\mathbf{k})$ is different from zero only at the points $\mathbf{k} = \mathbf{k}_0, -\mathbf{k}_0$; however, if one considers the equation for $\Phi_{ij}(\mathbf{k};t)$ (5) from ([2]) and (16), one can see that, with time, the values of $\Phi_{ij}(\mathbf{k};t)$ will become nonzero at the points $\mathbf{k} = 2\mathbf{k}_0; -2\mathbf{k}_0; 3\mathbf{k}_0; -3\mathbf{k}_0 ...$, which corresponds to the transfer of energy to waves with multiple wave numbers.

If one considers the interaction of a finite number of waves, the problem reduces —as for equations (5), (6) ([2]), (16), so also for system (3)—to the solution of a system of ordinary differential equations of the first order, and admits a numerical solution on a computer. The solution of equation (5) with the initial condition

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$V(\mathbf{k};0) = \sum_{-N\le n\le N;\,-M\le m\le M} \{D_{mn}[\delta(\mathbf{k}-\mathbf{k}_{mn}) + \delta(\mathbf{k}+\mathbf{k}_{mn})] + + iB_{mn}[\delta(\mathbf{k}-\mathbf{k}_{mn}) - \delta(\mathbf{k}+\mathbf{k}_{mn})]\}; \quad \mathbf{k}_{mn} = (mk_1^0;\, nk_2^0), \tag{18}$$

in which the interaction of $N \cdot M$ waves is taken into account, was obtained on the M-20 computer. The results of the numerical analysis of (5), (18) confirm the conclusion of item I.

III. In Figs. 1 and 2 ($k_1^0 = 1;\ k_2^0 = 5;\ \mathrm{Re} = 10^4;\ A = 0.002$) there is shown (curves 2) the change with time of the energy of two waves $\mathbf{k}_0$ and $2\mathbf{k}_0$, obtained

in the numerical solution of the system of equations (5), (6), (2), (16), with the initial conditions (17). The decrease of $E_1(\mathbf{k}_0;t)$, $E_1(2\mathbf{k}_0;t)$—the energy of the longitudinal component of the pulsation velocity of the waves $\mathbf{k}_0$ and $2\mathbf{k}_0$—to zero at $t^* = k_2^0/k_1^0$ is due to the fact that at this moment the direction of the vector $\delta\mathbf{V}(\mathbf{x},t)$

Fig. 1

is perpendicular to the x_1 axis; $\mathbf{k}_0 = (k_1^0;k_2^0)$. Curves **1** show the change in the energy of these waves obtained by solving the system (3), (14). Curves **3** represent the results of solving the closed system of equations (5), (6), (7), (1) from (2), which takes into account fifth moments. For small A, the

Fig. 2

qualitative agreement of the calculation results is very good; the quantitative agreement worsens as the initial amplitude A increases.

IV. The conclusion obtained in the present work on the instability of a flow with a gradient with respect to vortex disturbances of finite amplitude and on the decrease of the "critical amplitude"with increasing Reynolds number confirms the concept of the onset of turbulence of Acad. A. N. Kolmogorov (seminar on selected questions of analysis, Moscow State University, 1958).

Received
5 VIII 1964

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REFERENCES

1. E. M. Khazen, DAN, **147**, No. 1 (1962).
2. E. M. Khazen, DAN, **153**, No. 5 (1963).

Note: Figure translations are in progress. See original paper for figures.