M. D. RAMAZANOV

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Abstract

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M. D. RAMAZANOV

A PRIORI ESTIMATES OF L^p-TYPE FOR SOLUTIONS OF PARABOLIC EQUATIONS

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MATHEMATICS

The paper studies a class of function spaces close in their properties to the spaces L^p and denoted below by $V_{p,x}$, which have proved convenient in describing general boundary-value problems for parabolic equations.

§ 1. Let $f(x) = f(x_1, \dots, x_n)$ be a finite sufficiently smooth function in E^n, and let $\tilde{f}(\vec{\alpha}) = \tilde{f}(\alpha_1, \dots, \alpha_n)$ be its Fourier transform. The Euclidean space formed by the set $(\alpha_1, \dots, \alpha_n)$ of real arguments of the function $\tilde{f}(\vec{\alpha})$ will be denoted by R^n. We divide the first octant of the space R^n—the set $\Pi_{j=1}^n(0 < \alpha_j < \infty)$—by the hyperplanes $\alpha_j = 2^{m_j}$, $m_j = 0, \pm1, \pm2, \dots$ $(j = 1, \dots, n)$, into parallelepipeds $\Pi(\mathbf{m}) = \Pi(m_1, \dots, m_n)$, and associate with each parallelepiped $\Pi(\mathbf{m})$ the function $f_m^{(1)}(x)$, whose Fourier transform $\tilde{f}_m^{(1)}(\vec{\alpha})$ coincides on $\Pi(\mathbf{m})$ with the function $\tilde{f}(\vec{\alpha})$ and is equal to zero in $R^n \setminus \Pi(\mathbf{m})$. Carrying out similar constructions for each j-th octant $(j = 1, \dots, 2^n)$, we associate with the function $f(x)$ the vector-valued function with a countable number of components $F(x) = \{f_m^{(j)}(x)\}$; here $j = 1, \dots, 2^n$ is the number of the octant, $\mathbf{m} = (m_1, \dots, m_n)$, $m_k = 0, \pm1, \pm2, \dots$.

It can be shown that, for a finite sufficiently smooth function $f(x)$, the quantity

$$\|f\| = \sum_{m,j} \|f_m^{(j)}(x)\|_{L^p(x)} = \|\|F(x)\|_{L^p(x)}\|_{l^1} \quad (1 < p < \infty) \tag{1}$$

is finite.

The closure of finite sufficiently smooth functions in the norm (1) forms a Banach space, which we denote by $V_{p,x}(E^n)$, and the norm (1) by $(f \mid V_{p,x}(E^n))$. Since $f(x) = \sum_{m,j} f_m^{(j)}(x)$, we have

$$\|f\|_{L^p} = \left\| \sum_{m,j} f_m^{(j)}(x) \right\|_{L^p} \leq \sum_{m,j} \|f_m^{(j)}(x)\|_{L^p} = (f \mid V_{p,x}(E^n)),$$

i.e., the embedding $V_{p,x}(E^n) \subset L^p$ holds. It can be shown that $V_{p,x}(E^n) \neq L^p$.

The relations between the Sobolev spaces $W^k_{p,x}$ (1,2) and the spaces $V_{p,x}(E^n)$ are established by the following theorem.

Theorem 1. Let $f(x)$ be a finite function with support of diameter a. Then for any $\varepsilon > 0$ and $1 < p < \infty$ the inequality

$$(f \mid V_{p,x}(E^n)) \le K(p,\varepsilon)[\ln(a+2)]^n \|f\|_{W^{\varepsilon}_{p,x}}$$

holds with a constant $K(p,\varepsilon)$ independent of the function $f(x)$.

To characterize the differential properties of functions, we now define spaces with derivatives by means of Bessel potentials (3), starting from the space $V_{p,x}(E^n)$. Namely, we shall say that a function $f(x)$ belongs to the space

$$V^k_{p,x}(E^n) = V^{k_1,\dots,k_n}_{p,x_1,\dots,x_n}(E^n), \quad k_j \ge 0,$$

if there belongs to the space $V_{p,\mathbf{x}}(E^n)$ a function $g(\mathbf{x})$ whose Fourier transform is defined by the equality

$$\tilde{f}(\vec{\alpha}) \sum_{j=1}^{n} (|\alpha_j|^{k_j} + 1) = \tilde{g}(\vec{\alpha}).$$

As the norm of the function $f(\mathbf{x})$ in the space $V^{\mathbf{k}}_{p,\mathbf{x}}(E^n)$ we take the quantity

$$(f \mid V^{\mathbf{k}}_{p,\mathbf{x}}(E^n)) = (g \mid V_{p,\mathbf{x}}(E^n)).$$

For the spaces $V^{\mathbf{k}}_{p,\mathbf{x}}(E^n)$ there hold exact theorems on the trace of a function on the subspace E^{n-1} and on the extension of a function from the subspace E^{n-1} to all of E^n.

Theorem 2. If the function $f(\mathbf{x}) \in V^{\mathbf{m}}_{p,\mathbf{x}}(E^n)$ and $m_n \ge 1/p$, then the function $f(\mathbf{x}', 0) \in V^{\mathbf{k}'}_{p,\mathbf{x}'}(E^{n-1})$, where $\mathbf{x}' = (x_1, \dots, x_{n-1})$, $\mathbf{k}' = (k_1, \dots, k_{n-1})$, $k_j = m_j(1 - 1/pm_n)$, $j = 1, \dots, n-1$.

Theorem 3. Suppose that on the hyperplane $x_n = 0$ a function $g(\mathbf{x}') \in V^{\mathbf{k}'}_{p,\mathbf{x}'}(E^{n-1})$ is given. Then for every $l > 0$ there exists a function $f_l(\mathbf{x})$, which is defined in all of E^n, coincides on the hyperplane $x_n = 0$ with the function $g(\mathbf{x}')$ in the sense of the space $V^{\mathbf{k}'}_{p,\mathbf{x}'}(E^{n-1})$,

$$f_l(\mathbf{x}) \in V^{\mathbf{m}'}_{p,\mathbf{x}}(E^n), \qquad \text{where } m_j = k_j\left(1 + \frac{1}{pl}\right), \; j = 1, \dots, n-1, \quad m_n = l + \frac{1}{p}.$$

Theorems of this type have been proved in the spaces $W^{\mathbf{k}}_{2,\mathbf{x}}(E^n)$ (2), $H^{\mathbf{k}}_{p,\mathbf{x}}(E^n)$ (4), $B^{\mathbf{k}}_{p,\theta,\mathbf{x}}(E^n)$ (5). At the same time it is known (5) that in the spaces $W^{\mathbf{k}}_{p,\mathbf{x}}(E^n)$

for $p \neq 2$ exact theorems on traces and extensions of functions cannot be established.

We shall call a function $\Phi(\vec{\alpha})$ a **multiplier from the space $V_{p,\mathbf{x}}(E^n)$ to the space** $V_{q,\mathbf{x}}(E^n)$ if the operator $Af = F^{-1}\Phi F f$, acting from $V_{p,\mathbf{x}}(E^n)$ into $V_{q,\mathbf{x}}(E^n)$, is bounded. Here F is the Fourier-transform operator, and Φ is the operator of multiplication by the function $\Phi(\vec{\alpha})$.

Theorem 4. Suppose that the function $\Phi(\vec{\alpha})$ is continuous together with all derivatives of the form $\partial^k\Phi/\partial\alpha_{j_1}\dots\partial\alpha_{j_k}$, where $j_l \neq j_m$ for $l \neq m$; $k = 1, \dots, n$, outside the coordinate planes $\alpha_s = 0$ $(s = 1, \dots, n)$, and suppose that the inequalities

$$\left| \alpha_1^{k_1+\beta} \cdots \alpha_n^{k_n+\beta} \frac{\partial^k \Phi}{\partial\alpha_1^{k_1} \cdots \partial\alpha_n^{k_n}} \right| \leq C,$$

are satisfied, where

$$\beta = \frac{1}{p} - \frac{1}{q} \geq 0;$$

$k_s = 0$ or 1, $k_1 + \dots + k_n = k$ takes the values $0, 1, \dots, n$. Then the function $\Phi(\vec{\alpha})$ is a multiplier from the space $V_{p,\mathbf{x}}(E^n)$ to the space $V_{q,\mathbf{x}}(E^n)$.

The corresponding result for multipliers from L^p to L^q was established in (^{6}).

We shall also need the following

Theorem 5. Suppose that the function $f(\mathbf{x}) \in V_{p,\mathbf{x}}(E^n)$. Then the function $c(\mathbf{x}) \cdot f(\mathbf{x}) \in V_{p,\mathbf{x}}(E^n)$, if $c(\mathbf{x})$ is finite and satisfies the conditions

$$\max_{x_{j_{k+1}},\dots,x_{j_n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \tilde{c}(\alpha_{j_1}, \dots, \alpha_{j_k}; x_{j_{k+1}}, \dots, x_{j_n}) \right| \times$$

$$\times \prod_{l=1}^{k} \ln(|\alpha_{j_l}| + 2)\, d\alpha_{j_1} \cdots d\alpha_{j_k} \leq C$$

for all possible sets $(j_1 \dots j_k)$, $k = 0, \dots, n$. Here $c(\alpha_{j_1}, \dots, \alpha_{j_k}; x_{j_{k+1}}, \dots, x_{j_n})$ denotes the Fourier transform of the function $c(x_1, \dots, x_n)$ with respect to the variables $x_{j_1}, \dots, x_{j_k}$.

§ 2. Consider the space E^{n+1}; a point of this space will be denoted by $(t, \mathbf{x})$. Let Q be a compact domain in E^{n+1}, contained between the planes $t = 0$ and $t = T$ and bounded by a smooth surface S. We assume that for each $t \in [0, T]$ the set $\omega_t = Q \cap (t = \text{const})$ is a simply connected domain with smooth boundary $S \cap (t = \text{const})$, and that the surface S is such that at every point at which the normal to S is parallel to the axis Ot, the tangent plane has contact with the surface S of order no higher than some number θ, fixed for the given domain Q. We shall call θ the order of singularity of the surface S.

In the domain Q consider the parabolic equation

$$Lu \equiv u_t + \sum_{|k| \leq 2m} a_k(t, \mathbf{x}) D_x^k u = f(t, \mathbf{x}),$$

$$\operatorname{Re} \sum_{|k|=2m} a_k(t, \mathbf{x})(i\vec{\alpha})^k \geq \delta \sum_{j=1}^{n} \alpha_j^{2m} \tag{2}$$

for all real α_j, $j = 1, \dots, n$, with a constant $\delta > 0$, the same for all $(t, \mathbf{x}) \in Q$. Here $k = (k_1, \dots, k_n)$, k_j are integers, $k_j \geq 0$, $|k| = k_1 + \dots + k_n$, $D_x^k u = D_{x_1}^{k_1} \cdots D_{x_n}^{k_n} u$, $(i\vec{\alpha})^k = (i\alpha_1)^{k_1} \cdots (i\alpha_n)^{k_n}$.

Let $u(t, \mathbf{x})$ be, in Q, a solution of (2) satisfying the boundary conditions

$$u(0, \mathbf{x}) = \psi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \omega_0; \tag{3}$$

$$B_j(t, \mathbf{x}, D_t, D_x)u\big|_S = \varphi_j(t, \mathbf{x})\big|_S \quad (j = 1, \dots, m),$$

where $B_j(t, \mathbf{x}, D_t, D_x)$ is a linear differential operator with variable coefficients of order ν_j (with derivatives in t counted with weight $2m$). We shall assume that the operators B_j satisfy the condition of linear independence modulo the principal part of the operator L. A precise formulation of this condition can be found, for example, in [7].

The coefficients of the equation and of the boundary operators are assumed to be sufficiently smooth functions.

For a function f given in the domain Q, a function φ given on the surface S, and a function ψ given on ω_0, introduce the following norms:

$$(f \mid V_{p,t,\mathbf{x}}^{k_0,\mathbf{k}}(Q)) = \inf_g (g \mid V_{p,t,\mathbf{x}}^{k_0,\mathbf{k}}(E^{n+1})), \quad \text{where } g = f \text{ in } Q;$$

$$(\varphi \mid V_{p,t,s}^{k_0,k}(S)) = \inf_g (g \mid V_{p,t,\mathbf{x}}^{k_0 + k_0/pk,\ k+1/p}(E^{n+1})), \quad \text{where } g = \varphi \text{ on } S;$$

$$(\psi \mid V_{p,\mathbf{x}}^{k}(\omega_0)) = \inf_g (g \mid V_{p,t,\mathbf{x}}^{k/2m+1/p,\ k+2m/p}(E^{n+1})), \quad \text{where } g = \psi \text{ on } \omega_0.$$

A priori estimate theorem. Let a sufficiently smooth function $u(t, \mathbf{x})$ satisfy equation (2) and the boundary conditions (3). Then for any $\varepsilon > 0$ and $\gamma \geq \max(2m, \nu_j + 1/p)$ the estimate

$$(u \mid V_{p,t,\mathbf{x}}^{\gamma/2m,\gamma}(Q)) \leq K^{(\varepsilon)} \Big\{(f \mid V_{p,t,\mathbf{x}}^{\gamma+\varepsilon/2m-1,\ \gamma+\varepsilon-2m}(Q)) + \tag{4}$$

$$+(\psi \mid V_{p,\mathbf{x}}^{\gamma+\varepsilon-2m/p}(\omega_0)) + \sum_{j=1}^{m} \left(\varphi_j \mid V_{p,t,s}^{\frac{\gamma+\varepsilon-\nu_j-1/p}{2m},\ \gamma+\varepsilon-\nu_j-1/p}(S)\right)\Big\},$$

where the constant K does not depend on the function $u(t,\mathbf{x})$, if θ, the order of singularity of the surface S, satisfies the conditions:

$$\theta < 2m \quad \text{when} \quad \frac{\gamma}{2m} \leq \frac{1}{p} - \frac{1}{2m};$$

$$0 < \frac{1}{\gamma/2m - 1/p} \quad \text{when} \quad \frac{1}{p} + \frac{1}{2m} \leqslant \frac{\gamma}{2m} < 1 + \frac{1}{p}; \qquad \theta = 1 \quad \text{when} \quad 1 + \frac{1}{p} \leqslant \frac{\gamma}{2m}.$$

For a given γ one can precisely calculate the smoothness of the coefficients of the equation and of the boundary operators, and the smoothness S, so that inequality (4) is satisfied.

The proof of the theorem is based on the properties of the spaces $V_{p,x}^{k}$ established in Theorems 1–5.

Theorem 1 shows that from formula (4) one can obtain an a priori estimate in the norms $W_{p,x}^{k}$ of Sobolev spaces with arbitrary ε-accuracy.

For the straight cylinder, a priori estimates of the solution were obtained in papers (^{8},^{9},^{10}): in the norms $W_{2,t,x}^{\gamma/2m,\gamma}$ of the spaces (^{8}), in the norms $W_{2,t,x}^{k/2m,k}$ for k a multiple of $2m$ (^{9}), and in the norms $W_{p,t,x}^{\gamma/2m,\gamma}$ for any $\gamma \geqslant 2$, but for an equation of second order (^{10}).

A parabolic equation in domains with boundaries having singular points was first considered in (^{11}), where, for the heat-conduction equation, a classical solution of the first boundary-value problem was constructed.

For a general parabolic equation, the first boundary-value problem in domains having singularities on the boundary was studied in (^{12},^{13}).

Taking in formula (1) the outer norm of the function $F(x)$ in the space l^q ($1 \leqslant q \leqslant \infty$), one can arrive at new spaces $l^q(L^q)$. We have considered in detail the space $l^1(L^p) \equiv V_p(E^n)$; however, similar results are also established in the spaces $l_p(L^q)$ for any $1 \leqslant q \leqslant \infty$, and for parabolic equations a priori estimates analogous to those formulated above are obtained.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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