FREQUENCIES OF FREE VIBRATIONS OF RECTILINEAR CHAINS

1965

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Abstract

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MATHEMATICAL PHYSICS

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FREQUENCIES OF FREE VIBRATIONS OF RECTILINEAR CHAINS

(Presented by Academician A. A. Lebedev on 31 XII 1964)

At the present time, the problem of vibrations of rectilinear chains is of great interest. Usually the frequencies of small free vibrations of chains of identical atoms or of diatomic molecules are calculated using periodicity conditions ([1,2]). Sometimes solutions are found for exact boundary conditions ([3]).

To determine the frequencies of free vibrations of diatomic, triatomic, or, in the general case, k-atomic molecules, we introduced systems of polynomials ([2])

$$^{k}n_{km+l}(a_1, a_2, \dots, a_k, \alpha_1, \alpha_2, \dots, \alpha_k), \tag{1}$$

$$l = 1, 2, \dots, k; \quad m = 0, 1, 2, \dots,$$

satisfying the recurrence relation

$$^{k}n_{km+l} - f_{1k}(a_1, a_2, \dots, a_k, \alpha_1, \alpha_2, \dots, \alpha_k)\, ^{k}n_{(k-1)m+l} +$$

$$+ f_{2k}(a_1, a_2, \dots, a_k)\, ^{k}n_{(k-2)m+l} = 0. \tag{2}$$

For $k = 1,\ a_1 = 2x,\ \alpha_1 = -1$, the polynomials (1) coincide with the Gegenbauer polynomials $C_n^1(x)$ ([1]).

Let us determine the frequencies of free vibrations of a rectilinear chain of diatomic molecules $M_1 M_2 M_1 \dots,\ \alpha_1 \alpha_2 \alpha_1 \dots$, in which both ends are fixed (Fig. 1, I).

The equations of motion for free longitudinal vibrations with small amplitudes about positions of stable equilibrium have the form

$$M_1 \ddot{U}_1 = \alpha_1 (U_2 - U_1) - \alpha_2 U_1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$M_1\ddot{U}_{2n+1} = \alpha_1(U_{2n+2} - U_{2n+1}) + \alpha_2(U_{2n} - U_{2n+1}),$$

$$M_2\ddot{U}_{2n+2} = \alpha_2(U_{2n+3} - U_{2n+2}) + \alpha_1(U_{2n+1} - U_{2n+2}), \tag{3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$M_2\ddot{U}_N = \alpha_1(U_{N-1} - U_N) - \alpha_2 U_N.$$

Substituting into (3) the particular solution

$$U_n = |U_n| e^{i\omega t},$$

we obtain the equation for determining the frequencies

$$^2n_N(a_1, a_2, \alpha_1, \alpha_2) = 0, \tag{4}$$

where $a_1 = -M_1\omega^2 + \alpha_1 + \alpha_2,\ a_2 = -M_2\omega^2 + \alpha_1 + \alpha_2$.

For the polynomials $^2n_N(a_1, a_2, \alpha_1, \alpha_2)$, the recurrence relations (2) have the form

$$^2n_N - (a_1a_2 - \alpha_1^2 - \alpha_2^2)\, ^2n_{N-2} + \alpha_1^2\alpha_2^2\, ^2n_{N-4} = 0.$$

Let us introduce a quantity γ such that

$$a_1a_2 - a_1^2 - a_2^2 = 2a_1a_2 \operatorname{ch} 2\gamma; \tag{5}$$

then equation (4) can be reduced to the form

$$\frac{1}{\operatorname{sh} 2\gamma}\left[\operatorname{sh}(N+2)\gamma + \frac{a_2}{a_1}\operatorname{sh} N\gamma\right] = 0. \tag{6}$$

If one end is fixed and the other is free (Fig. 1, *II*), then

$$\frac{1}{\operatorname{sh} 2\gamma}\left[\operatorname{sh}(N+2)\gamma + \frac{a_2 - a_1}{a_1}\operatorname{sh} N\gamma\right] = 0; \tag{7}$$

if both ends are free (Fig. 1, *III*), then

$$\frac{\operatorname{sh} N\gamma}{\operatorname{sh} 2\gamma}\left[a_1a_2 - a_2(a_1+a_2) - a_1^2 + a_2^2\right] = 0. \tag{8}$$

Fig. 1. Rectilinear chain of diatomic molecules

In equations (7) and (8), γ, as in equation (6), is determined by relation (5).

For the case in which both ends are fixed, or one is fixed and the other is free, we obtain the following equation for determining the frequencies:

$$\omega^4 - \omega^2\frac{(M_1+M_2)(a_1+a_2)}{M_1M_2} + \frac{2a_1a_2}{M_1M_2}(1-\operatorname{ch} 2\gamma) = 0, \tag{9}$$

where γ are the roots of equations (6) and (7).

For a chain with free ends, equation (8) is solved exactly, and for the frequencies one obtains the values

$$\omega^2 = \frac{(M_1+M_2)(a_1+a_2)}{2M_1M_2} \pm \left\{\frac{\left[(M_1+M_2)(a_1+a_2)\right]^2}{4M_1^2M_2^2} - \frac{2a_1a_2}{M_1M_2}\left(1-\cos^2\frac{k_2\pi}{N}\right)\right\}^{1/2},$$

$$\omega^2 = \alpha_1(M_1+M_2)/M_1M_2, \qquad k_2 = 1, 2, \dots, N/2-1, \tag{10}$$

$$\omega^2 = 0.$$

The calculation of the vibration frequencies of a chain of triatomic molecules is carried out in an analogous manner; therefore we give at once the final form of the equations for γ:

$$\frac{1}{\operatorname{sh} 3\gamma}\left[\operatorname{sh}(N+3)\gamma + \frac{a_2a_3}{a_1a_2}\operatorname{sh} N\gamma\right] = 0, \tag{11}$$

if both ends are fixed;

$$\frac{1}{\operatorname{sh} 3\gamma}\left[\operatorname{sh}(N+3)\gamma + \frac{a_2a_3 - a_1a_2 + a_1^2}{a_1a_2}\operatorname{sh} N\gamma\right] = 0, \tag{12}$$

if the first end is fixed and the second is free;

$$\frac{\operatorname{sh} N\gamma}{\operatorname{sh} 3\gamma}\left[a_1a_2a_3 - a_3(a_1a_2+a_2a_3) - a_1a_2^2 - a_2a_3^2 - a_3a_1^2 + a_1^2a_3 + a_2^2a_3\right] = 0, \tag{13}$$

if both ends are free.

Having determined the quantity γ from (11), (12), and (13) and substituting its value into the equality

$$a_1a_2a_3 - a_1a_2^2 - a_2a_3^2 - a_3a_1^2 = 2a_1a_2a_3 \operatorname{ch} 3\gamma, \tag{14}$$

we obtain the corresponding frequency values.

Received
24 XII 1964

CITED LITERATURE

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3. M. Parodi, *Mém. Sci. Phys. Paris*, F. 47 (1944).

Note: Figure translations are in progress. See original paper for figures.