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Abstract

Full Text

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A NEW D -THEOREM IN THE THEORY OF FINITE GROUPS

(Presented by Academician I. M. Vinogradov, 1 VII 1964)

§ 1. In this note we use the same notation as in ⁽¹⁾: G is a finite group of order (G) (all groups considered are finite); Π is some set of primes; m_Π is the greatest Π -divisor ⁽²⁾ of the natural number m ; S_Π is a subgroup of the group G of order $(G)_\Pi$. Properties of the group G : E_Π^θ has an S_Π -subgroup; $E_\Pi^\theta - G$ has an S_Π -subgroup possessing the property θ ; $D_\Pi - G$ has an S_Π -subgroup H , and for any Π -subgroup A of G there is an element $x \in G$ such that

$$x^{-1}Ax = A^x \leq H;$$

$D_\Pi^\theta - G$ has the properties E_Π^θ and D_Π .

Notation for group-theoretic properties: n -nilpotency, s -solubility, ss -super-solubility, d -dispersiveness in the sense of Ore (the group G is dispersive in the sense of Ore ⁽¹⁰⁾ if and only if in G and in every one of its factor groups there is an invariant Sylow subgroup corresponding to the greatest prime divisor of the order), z -the property of cyclicity of all Sylow subgroups of the group. $\Pi(m)$ denotes the set of all distinct prime divisors of the natural number m .

§ 2. In the present note the following will be proved.

Theorem A. *If K is such a normal divisor of the group G that K has the property E_Π^z , and G/K has the property D_Π^d , then G has the property D_Π^s .*

Two consequences can be obtained from this theorem.

Corollary 1. *If K is such a normal divisor of the group G that K has the property E_Π^z , and G/K has the property D_Π^{ss} , then G has the property D_Π^{ss} .*

Corollary 2. *If every factor of some series of normal divisors of the group G has the property E_Π^z , then G has the property D_Π^{ss} .*

§ 3. **Lemma 1.** *Let G have the property E_Π^z , and let M be any Π -subgroup of G . Then the normalizer N of the subgroup M in G has the property E_Π^z .*

Proof. Suppose that the lemma is true for all groups whose orders are less than (G) . By Theorem 16 of ⁽³⁾, G has the property D_Π , and therefore M is contained in some S_Π -subgroup H of G . Let $M \neq 1$. Since M , obviously, is supersolvable, it contains an invariant Sylow subgroup $P \neq 1$. It is clear that $N \leq N_1$, where N_1 is the normalizer of P in G . If P is invariant in G , then, by

induction in the group G/P , the normalizer N/P of the subgroup M/P has an S_Π -subgroup L/P . The subgroup L will be the desired one. Let $N_1 \neq G$, and let Q be a Sylow q -subgroup ($q \in \Pi$) of N_1 . Obviously, $P^x Q^x \leq H$ for some $x \in G$. By Theorem 16 of ⁽³⁾, P and P^x are conjugate in H , and hence the order of the normalizer of the subgroup P in H is divisible by (Q) . Hence, in view of the arbitrariness of q , it follows that N_1 has the property E_Π^z . Since $M \leq N \leq N_1$ and $N_1 \neq G$, it follows by induction that N has the property E_Π^z . The lemma is proved.

Lemma 2. *Suppose that Theorem A is valid in the case when $[G/K]$ is divisible by no more than one prime number from Π . Let $G = PK$, K invariant in G , $P \cap K = 1$, P is a p -group, where p is the greatest*

prime number from Π dividing $[G]$, and suppose that K has property E_{Π^z} . Then the normalizer N of the subgroup P in G has property E_Π .

Proof. Let H be an S_π -subgroup of G . By Lemma 1 from (1), $H \cap K = A$ is an S_Π -subgroup of K . Suppose that P is nontrivial and is contained in a Sylow p -subgroup H_p of H . The lemma is obvious if (A) is a power of a prime number. Therefore suppose that (A) is divisible by at least two distinct prime numbers.

We shall carry out the proof by induction on (G) and on $(G)_\Pi$.

Suppose H has a nilpotent $S_{p,q}$ -subgroup B , where $q \in \Pi((H))$, $q \neq p$. By induction, N has an $S_{\Pi q}$ -subgroup Q . Since G has property D_Π , we may assume that $PQ \subseteq H$. From this, taking into account the nilpotency of B , we obtain that the normalizer of the subgroup P in H contains Q and a Sylow q -subgroup of H , and therefore is an S^Π -subgroup of N . Hence let H satisfy the condition

I. The $S_{p,q}$ -subgroups of H are nonnilpotent for every element $q \in \Pi((H))$ distinct from p .

Let K be p -solvable ^(4,6). Then, obviously, G will be a p -supersolvable group, i.e., it has a chief series each of whose indices is either equal to p or is not divisible by p . Let ω be all those elements of Π with respect to which G is supersolvable, and let R be the maximal invariant ω' -subgroup of the group G , where $\omega' = \Pi((G)) \setminus \omega$. Taking into account Theorem XVII from ⁽⁴⁾, we see that the commutator subgroup of G/R is a Π -group. Hence G/R is solvable.

Suppose now that the order R is divisible by some number $q \in \Pi$, $q \notin \omega$. If $(R)_q = (G)_q$, then consider the intersection $R \cap B$, where B is an $S_{p,q}$ -subgroup of H . Since $p \in \omega$, we have $q \neq p$. Hence $R \cap B$ is a Sylow q -subgroup of B , invariant in it. Since by hypothesis $p > q$, B will also have an invariant Sylow p -subgroup, i.e. B is nilpotent, contrary to I. If, however, $(R)_q < (G)_q$, then, by Theorem 4.2 from ⁽⁵⁾, the group G is q -supersolvable, which is impossible in view of $q \notin \omega$.

Thus G/R is a solvable group of order divisible by $(G)_\Pi$. Hence the subgroup N is Π -solvable and, by Theorem 2 from ⁽⁶⁾, has property E_Π . In view of this, suppose that K satisfies the condition.

II. K is not p -solvable.

Let A_p be a Sylow p -subgroup of A . In view of II, $(A_p) \neq 1$. Taking into account the condition of the lemma, we see that A_p is invariant in H . Let P_1 be a subgroup of order p in A_p . Since A_p is cyclic, P_1 is invariant in H and is the unique subgroup of order p in A . Let P_0 be the normalizer of P in H_p . It is clear that $P \neq P_0$. By Dedekind's identity we have $P[P_0 \cap K] = P_0$. Obviously, $P_0 \cap K \subseteq A$ and $(P_0 \cap K) > 1$. Since P_1 is the unique subgroup of order p in A , we have $P_1 \subseteq P_0 \cap K \subseteq P_0 \subseteq N$.

Suppose now that (A) is divisible by at least three distinct prime numbers. Then one can form such sets σ and τ that

$$\Pi = \sigma \cup \tau, \quad \sigma \cap \tau = p, \quad \sigma \cap \Pi((A)) \neq p, \quad \tau \cap \Pi((A)) \neq p.$$

By induction, N has an S_σ -subgroup U and an S_τ -subgroup V . Obviously, P is contained in U and V . Since $P_1 \subseteq N$, assume, without loss of generality, that P_1 is contained in U and V . Since G has property D_Π , $U^x \subseteq H$, $V^y \subseteq H$, where $x, y \in G$. Therefore P_1^x and P_1^y are contained in H , and hence also in A . Since P_1 is the unique subgroup of order p in A , $P_1 = P_1^x = P_1^y$. Taking this into account, we see that P_1 is invariant in U and in V , i.e. $U \subseteq N_1$ and $V \subseteq N_1$, where N_1 is the normalizer of P_1 in G . Moreover, $H \subseteq N_1$. By Dedekind's identity, $N_1 = P[N_1 \cap K]$. Since $A \subseteq N_1$, $N_1 \cap K$ has property E_{Π^z} . By induction the normalizer N' of the subgroup P in N_1 has property E_Π (by Theorem 4.2 from ⁽⁵⁾ and condition II, $N_1 \neq G$). Since U and V are contained in N_1 , it follows from this that N also has property N_Π .

Finally, let $(A) = p^m q^n$, where $q \in \Pi$, $q \neq p$, and let S and T be, respectively, a Sylow p - and a q -subgroup of $N \cap K$. We may assume, in view of

property D_Π for G , that $PT \subseteq H$. Moreover, $PS \subseteq H^x$ for some $x \in G$, and hence $S \subseteq A_p^x$, where A_p is a Sylow p -subgroup of A . Since $H_p^x = P^x A_p^x = P A_p^x$, S lies in the center of the subgroup H_p^x . Consequently, $S^{x^{-1}}$ lies in the center of the subgroup H_p , i.e. $S^{x^{-1}} \subseteq N$. Moreover, $S^{x^{-1}} \subseteq A_p$. Taking into account that A_p is a cyclic invariant subgroup of A , we see that $S^{x^{-1}} T$ is an S_Π -subgroup of $N \cap K$. Hence, $PS^{x^{-1}} T$ is an S_Π -subgroup of N , since $N = P[N \cap K]$.

The lemma is proved.

§ 4. We shall prove Theorem A by induction on (G) and $(G)_\Pi$, according to the scheme proposed by P. Hall for theorems of this type (see ⁽¹⁾, D5).

By a theorem of S. A. Rusakov ⁽³⁾, K has the property D_Π . Hence, by Theorem 1 of S. A. Chunikhin ⁽⁷⁾ (see also ⁽¹⁾), G has a solvable S_Π -subgroup H , and all S_Π -subgroups in G are conjugate. Therefore it remains for us to prove that every maximal Π -subgroup of the group G is its S_Π -subgroup. Let L be an arbitrary maximal Π -subgroup of G . By Lemma 1 of ⁽¹⁾, HK/K is an S_Π -subgroup of G/K , and $H \cap K$ is an S_Π -subgroup of K . According to the hypothesis, HK/K is dispersed by Ore, and all Sylow subgroups of $H \cap K$ are cyclic. Since G/K

has the property D_Π , we have $L \subseteq KH^x$ for some $x \in G$. If $KH^x \neq G$, then, by induction, KH^x has the property D_Π , and hence $(L) = (H)$. Therefore

I. $KH = G$.

Since, according to I, $G = K LH$, we have

$$(G) = \frac{(KL)(H)}{(KL \cap H)},$$

whence $(KL : KL \cap H) = (G : H)$. Therefore, if $KL \neq G$, then, by induction, $L^x \subseteq KL \cap H$ for some $x \in G$. Hence $L^x = H$. Therefore

II. $KL = G$.

Let $\omega = \Pi((L)) \neq \Pi((H))$. In view of Theorem D3 of ⁽¹⁾, we have: K has the property E_ω^z , G/K has the property D_ω^d . Since $(G)_\omega < (G)_\Pi$, by induction G has the property D_ω , and hence $L \subseteq H_\omega^x \subseteq H^x$, where H_ω is an S_ω -subgroup of H , $x \in G$. Hence, in view of the maximality of the subgroup L , it follows that $L = H^x$. Therefore we shall assume that the equality holds:

III. $\Pi((L)) = \Pi((H))$.

Let G have a minimal invariant Π -subgroup $X \neq 1$. If $X \cap K = 1$, then KX/X , isomorphic to K , is invariant in G/X , and moreover

$$G/X / KX/X \simeq G/K / KX/K.$$

Thus, G/X and KX/X satisfy the hypothesis of the theorem and, consequently, G/X has, by induction, the property D_Π . Then, by Lemma 4 of ⁽¹⁾, G has the property D_Π . If, however, $X \cap K \neq 1$, then $X \subseteq K$. Considering G/X and K/X , with induction and Lemma 4 of ⁽¹⁾ taken into account, we again arrive at the conclusion that G has the property D_Π . Therefore

IV. G has no invariant Π -subgroups distinct from the identity.

Suppose now that $L \cap K = M \neq 1$. Since all Sylow subgroups of M are cyclic, M has an invariant Sylow p -subgroup $M_p \neq 1$, $p \in \Pi$. Let \bar{N} be the normalizer of M_p in G . Obviously, $L \subseteq \bar{N}$. By Lemma 1 the subgroup $N = \bar{N} \cap K$ has the property E_Π^z . Moreover, in view of II,

$$\bar{N}/N \simeq K\bar{N}/K = G/K.$$

Since, in view of IV, $\bar{N} \neq G$, by induction \bar{N} has the property D_Π , and therefore $(L) = (\bar{N})_\Pi$. By Lemma 1 of ⁽¹⁾, in view of the equality

$$M = [L \cap \bar{N}] \cap K = L \cap N,$$

the subgroup M is an S_Π -subgroup of N . Hence $(M_p) = (N)_p$. From this, by virtue of the normalizer property of a p -group, it follows that $(M_p) = (K)_p$. Since K has the property D_Π , M is contained in some S_Π -subgroup S of K . In

view of Lemma 1 of ⁽¹⁾, suppose that $M_p \subseteq M \subseteq S = H \cap K$. Hence, taking Frattini's lemma into account (see ⁽⁸⁾ or ⁽⁹⁾, Lemma 1), we obtain:

$$[\overline{N} \cap H]S = H.$$

Thus,

$$\frac{(\overline{N} \cap H)(S)}{(\overline{N} \cap S)} = (H).$$

But $\overline{N} \cap S = N \cap S = M$, since M is an S_Π -subgroup

group from N . Therefore $(\overline{N} \cap H) = \frac{(H)}{S} \cdot (M)$. But from II it follows that $(L) = (G/K)_\Pi(M)$. Consequently, $(\overline{N} \cap H) = (L)$. Recalling that \overline{N} has the property D_Π , we obtain $[\overline{N} \cap H]^x = L$, where $x \in G$. Hence, $L \subseteq H^x$, as was required.

Let $L \cap K = 1$. Taking into account Sylow's theorem, we are convinced of the validity of the following assertion:

V. Theorem A is true if (G/K) is divisible by no more than one prime from Π .

We shall also assume that $K \neq G$, since otherwise the theorem is true.

Let p be the greatest prime divisor of the number $(G)_\Pi \neq 1$. In view of the condition of the theorem and of remarks II and III, the group G/K has an invariant Sylow subgroup T/K of order p^m , $m > 0$. Let $P = L \cap T$. Since $G = KL = LT$ and $L \cap K = 1$, we have $(P) = p^m$, i.e., P is a Sylow subgroup of L , invariant in it. Let \overline{Q} be the normalizer of P in G . Obviously, $L \subseteq \overline{Q}$. Since $T = PK$, $P \cap K = 1$, assertion V allows us to apply Lemma 2, according to which $\overline{Q} \cap T$ has the property E_Π . Hence, in view of Lemma 1 from ⁽¹⁾, $Q = \overline{Q} \cap K$ has the property E_{Π^z} . Since, by Dedekind's relation, $\overline{Q} = L[\overline{Q} \cap K] = LQ$, we have $\overline{Q}/Q \simeq L \simeq LK/K = G/K$. According to IV, $\overline{Q} \neq G$. Consequently, by induction \overline{Q} has the property D_Π , and hence L is an S_Π -subgroup of the group \overline{Q} . From this, in view of the isomorphism $\overline{Q}/Q \simeq L$, it follows that $(Q)_\Pi = 1$. But then P will be a Sylow p -subgroup in T , and hence also in $[H \cap T]^x$. It is not difficult to note, by virtue of the property E_{Π^z} for K , that an S_Π -subgroup $H \cap T$ of the group T is supersolvable. Since p is the greatest prime divisor of (H) , it follows that P is invariant in $[H \cap T]^x$, and hence also in H^x . Thus, H^x and L are contained in \overline{Q} , which has the property D_Π . Consequently, $(L) = (H)$.

The theorem is completely proved.

§ 5. **Definition.** The group G is called $\pi\theta$ -closed if it has an invariant S_Π -subgroup possessing the property θ .

Using the constructions of the proofs of two theorems—theorem A and theorem D5 of Ph. Hall ⁽¹⁾—one can obtain the following, more general result.

Theorem B. Let Π consist of the nonintersecting subsets σ and τ . Let some normal divisor K of the group G have an S_Π -subgroup that is σn -closed and τz -

closed, and let G/K have the properties D_{Π^s} and E_{τ^d} . Then G has the property D_{Π^s} .

If σ is empty, then theorem A is obtained from theorem B. If, however, τ is empty, then from theorem B, as a special case, one obtains theorem D5 of the paper of Ph. Hall ⁽¹⁾.

In the proof of theorem B one uses theorem 16 of the paper of S. A. Rusakov ⁽³⁾ and the theorem of H. Wielandt ⁽¹¹⁾. The following also turns out to be useful.

Lemma 3. *Let G have a normal divisor K , possessing the property E_{Π^n} , and let M be an arbitrary solvable Π -subgroup of the group G . Then $N \cap K$ has the property E_{Π^n} , where N is the normalizer of the subgroup M in G .*

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