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A. Guichardet (France)

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Abstract

Full Text

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MATHEMATICS

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ON TENSOR PRODUCTS OF C^* -ALGEBRAS

(Presented by Academician P. S. Novikov on 21 IX 1964)

1. If A_1 and A_2 are C^* -algebras, then their algebraic tensor product is denoted by $A_1 \otimes A_2$; on $A_1 \otimes A_2$ one can define the norm

$$\|x\|_* = \sup_{\pi_1, \pi_2} \|(\pi_1 \otimes \pi_2)(x)\|,$$

where π_i is an arbitrary representation of A_i . This norm has been studied in (1-4); it has the following properties

$$\begin{aligned} \|xy\|_* &\leq \|x\|_* \|y\|_*, \\ \|xx^*\|_* &= \|x\|_*^2, \\ \|x_1 \otimes x_2\|_* &= \|x_1\|_* \|x_2\|_*, \\ \|x\|_* &= \|(\rho_1 \otimes \rho_2)(x)\|, \end{aligned} \tag{1}$$

where ρ_i is a faithful representation of A_i ; the completion of $A_1 \otimes A_2$ in this norm is a C^* -algebra, which we denote by $A_1 \otimes A_2$; it is known (2) that $\|\cdot\|_*$ is the least norm satisfying (1). Here another norm on $A_1 \otimes A_2$ will be defined and a new completion studied.

2. To every pair (π_1, π_2) of representations of A_1 and A_2 in one Hilbert space satisfying

$$\pi_1(x_1) \cdot \pi_2(x_2) = \pi_2(x_2) \cdot \pi_1(x_1), \quad x_i \in A_i,$$

one can associate a representation π of the algebra $A_1 \otimes A_2$ such that

$$\pi(x_1 \otimes x_2) = \pi_1(x_1) \cdot \pi_2(x_2), \tag{2}$$

$$\|\pi(x_1 \otimes x_2)\| \leq \|x_1\| \cdot \|x_2\|; \quad (3)$$

conversely, every representation of $A_1 \otimes A_2$ satisfying (3) is obtained in this way (this is proved as Proposition 1 in ⁽¹⁾).

Definition. For every $x \in A_1 \otimes A_2$, put

$$\|x\|_\nu = \sup_{\pi} \|\pi(x)\|,$$

where π is an arbitrary representation of $A_1 \otimes A_2$ satisfying (3).

Theorem 1.

- 1) The function $\| \cdot \|_\nu$ is the greatest norm on $A_1 \otimes A_2$ having properties analogous to (1).
- 2) If A_1 or A_2 is of type I, then $\| \cdot \|_\nu = \| \cdot \|_*$.

It is not hard to see that if

$$x = \sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in A_1 \otimes A_2,$$

then

$$\|x\|_* \leq \|x\|_\nu \leq \sum_{i=1}^n \|x_{1,i}\| \|x_{2,i}\|$$

and that $\| \cdot \|_\nu$ is a norm with properties (1).

Let now p be a norm with properties (1); realize the completion B with respect to the norm p in a Hilbert space; the inequality $p(x) \leq \|x\|_\nu$ follows from the fact that the natural mapping $A_1 \otimes A_2 \rightarrow B$ is a representation with properties (3). Assertion 2) follows from Theorem 3 ⁽²⁾. We shall denote by $A_1 \overset{\vee}{\otimes} A_2$ the C^* -algebra which is the completion of $A_1 \otimes A_2$ with respect to the norm $\| \cdot \|_\nu$; the identity mapping extends to a homomorphism $A_1 \overset{\vee}{\otimes} A_2$ onto $A_1 \overset{*}{\otimes} A_2$, which we shall call canonical.

Remark 1. It is not known whether the norms $\| \cdot \|_*$ and $\| \cdot \|_\nu$ always coincide; proving this would be equivalent to proving the following assertion: if $T_1, \dots, T_n, S_1, \dots, S_n$ are operators in a Hilbert space H and $S_{iT}j = T_{jS}i$, then

$$\left\| \sum_{i=1}^n S_{iT}i \right\| \leq \left\| \sum_{i=1}^n S_i \otimes T_i \right\|,$$

where $S_i \otimes T_i$ denotes an operator in $H \otimes H$. It is known only that this assertion is always valid when T_i belongs to some C^* -algebra A of type I and $S_i \in A'$.

3. Tensor products of homomorphisms. Let u_i , $i = 1, 2$, be a homomorphism of the algebra A_i into the C^* -algebra B_i ; the homomorphism $u_1 \otimes u_2$ of $A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ can be regarded as a representation with property (3) and therefore extended to a homomorphism

$$A_1 \overset{\vee}{\otimes} A_2 \rightarrow B_1 \overset{\vee}{\otimes} B_2.$$

Theorem 2. *Suppose that u_i maps A_i onto B_i , and denote its kernel by I_i ; then u has kernel $I_1 \overset{\vee}{\otimes} A_2 + A_1 \overset{\vee}{\otimes} I_2$. ($I_1 \overset{\vee}{\otimes} A_2$ and $A_1 \overset{\vee}{\otimes} I_2$ denote, respectively, the closures of $I_1 \otimes A_2$ and $A_1 \otimes I_2$ in $A_1 \overset{\vee}{\otimes} A_2$.)*

Let ω be the natural homomorphism $A \rightarrow A/I$; obviously $\text{Ker } u \supset I$; it remains to prove that

$$\|\omega(x)\| \leq \|u(x)\| \quad \text{for } x \in A_1 \otimes A_2. \quad (4)$$

If $x_i \in A_i$, then $\omega(x_1 \otimes x_2)$ depends only on $u_1(x_1)$ and $u_2(x_2)$. Put

$$\omega(x_1 \otimes x_2) = v(u_1(x_1), u_2(x_2));$$

v defines a homomorphism $v' : B_1 \otimes B_2 \rightarrow A/I$ with the property

$$v'(y_1 \otimes y_2) = v(y_1, y_2);$$

we shall show that

$$\|v'(y_1 \otimes y_2)\| \leq \|y_1\| \cdot \|y_2\| \quad \text{for } y_i \in B_i,$$

in other words, that

$$\|\omega(x_1 \otimes x_2)\| \leq \|u_1(x_1)\| \cdot \|u_2(x_2)\| \quad \text{for } x_i \in A_i.$$

It is enough to prove that

$$\|v'(y_1 \otimes y_2)\| \geq \|y_1\| \cdot \|y_2\| \quad \text{for } y_i \in B_i, \quad (5)$$

Since $x_1 \otimes x_2 - (x_1 + z_1) \otimes (x_2 + z_2) \in I$, it follows that

$$\|\omega(x_1 \otimes x_2)\| \leq \|(x_1 + z_1) \otimes (x_2 + z_2)\| = \|x_1 + z_1\| \cdot \|x_2 + z_2\|.$$

From (5) it follows that

$$\|v'(y)\| \leq \|y\|_\nu \quad \text{for } y \in B_1 \otimes B_2,$$

and hence (4) is readily obtained.

Remark 2. Up to now the analogous result has not been proved for $A_1 \overset{*}{\otimes} A_2$; on the other hand, it is known for $A_1 \overset{*}{\otimes} A_2$, but not for $A_1 \overset{\vee}{\otimes} A_2$, that the tensor product of two exact homomorphisms is also exact. From all this one may conclude that, if A_1 or A_2 is of type I, then $\text{Ker } u = I$, even when u_1 and u_2 are not mappings “onto.”

Remark 3. It is known ⁽²⁾ that if A_1 and A_2 are simple, then $A_1 \overset{*}{\otimes} A_2$ is also simple; this is unknown for $A_1 \overset{\vee}{\otimes} A_2$.

4. Representations of $A_1 \overset{\vee}{\otimes} A_2$. Let π_i be a representation of A_i in the Hilbert space H_i . The representation $\pi_1 \otimes \pi_2$ of the algebra $A_1 \overset{*}{\otimes} A_2$ in the space $H_1 \otimes H_2$ was studied in ^(1,4); if it is multiplied on the left by the canonical homomorphism $A_1 \overset{\vee}{\otimes} A_2 \rightarrow A_1 \overset{*}{\otimes} A_2$, one obtains a representation $\pi_1 \overset{\vee}{\otimes} \pi_2$ of the algebra $A_1 \overset{\vee}{\otimes} A_2$, which has some of the properties of $\pi_1 \overset{*}{\otimes} \pi_2$ (the same is true for tensor products of traces ⁽¹⁾); for example, $\pi_1 \overset{\vee}{\otimes} \pi_2$ is irreducible and completely continuous and only if π_1 and π_2 are irreducible and completely continuous ⁽¹⁾; if A_1 and A_2 are assumed separable, then the analogous assertion is true with respect to irreducibility and normality ⁽¹⁾.

It is proved, as in ⁽¹⁾, that every factor representation of finite type (respectively, every irreducible completely continuous representation) of $A_1 \overset{\vee}{\otimes} A_2$ is the tensor product of two representations with the same properties.

It follows from ^(1,4) that $A_1 \overset{\vee}{\otimes} A_2$ is CCR if and only if A_1 and A_2 are CCR; that if A_1 and A_2 are assumed separable, then the same is true with respect to the property GCR; finally, it is proved, as in ⁽¹⁾, that if A_1 and A_2 are separable and $A_1 \overset{\vee}{\otimes} A_2$ is NGCR, then A_1 or A_2 is NGCR; it is unknown whether the converse is true.

Remark 4 (examples of irreducible representations of $A_1 \overset{\vee}{\otimes} A_2$ that are not tensor products of representations). Let A_1 be a C^* -algebra with a factor representation ρ_1 of type II; ρ_1 is quasi-equivalent to a representation π_1 such that the corresponding factor B is anti-isomorphic to its commutant B' ; let Φ be an anti-isomorphism of B onto B' ; $\pi_2(x) = \Phi(\pi_1(x))$ is a representation of the algebra A_2 , opposite to A_1 ; the representation π of the algebra $A_1 \overset{\vee}{\otimes} A_2$, defined by equality (2), is irreducible, but is not quasi-equivalent to a tensor product of representations, since its restrictions to A_1 and A_2 are not of type I.

5. The C^* -algebra of the direct product of groups. Let G_1 and G_2 be locally compact groups, A_1 and A_2 their C^* -algebras, and A the C^* -algebra of the group $G = G_1 \times G_2$.

Theorem 3. *There exists an isomorphism of $A_1 \overset{\vee}{\otimes} A_2$ onto A which sends $x_1 \otimes x_2$ to the function on G : $(g_1, g_2) \mapsto x_1(g_1) \cdot x_2(g_2)$ (x_i is a continuous finite function on G_i).*

Denote by K_i the collection of continuous finite functions on G_i ; assigning to each element $x_1 \otimes x_2$, $x_i \in K_i$, the function on G :

$$(g_1, g_2) \mapsto x_1(g_1) \cdot x_2(g_2),$$

we obtain an isomorphism Φ of the algebra $K_1 \otimes K_2$ onto an everywhere dense subalgebra of the algebra A ; we must prove that

$$\|\Phi(x)\|_A = \|x\|_{\vee}$$

for every

$$x = \sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in K_1 \otimes K_2.$$

Consider all

pairs (π_1, π_2) of unitary representations of G_1 and G_2 satisfying the condition

$$\pi_1(g_1) \cdot \pi_2(g_2) = \pi_2(g_2) \cdot \pi_1(g_1).$$

For such a pair let $\pi_1 \times \pi_2$ denote the following representation

$$(\pi_1 \times \pi_2)(g_1, g_2) = \pi_1(g_1) \cdot \pi_2(g_2);$$

then

$$\|x\|_{\vee} = \sup \left\| \sum_{i=1}^n \pi_1(x_{1,i}) \cdot \pi_2(x_{2,i}) \right\| = \sup \|(\pi_1 \times \pi_2)(\Phi(x))\| = \|\Phi(x)\|_A.$$

Corollary. If the function 1 on G_1 (respectively on G_2) is the limit (in the topology of uniform convergence on every compact subset) of continuous finite positive definite functions, then the norms $\|\cdot\|_*$ and $\|\cdot\|_{\vee}$ on $A_1 \otimes A_2$ coincide.

Indeed, let l_1, l_2, l be the regular representations of G_1, G_2, G ; then

$$\|x\|_* = \|(l_1 \otimes l_2)(x)\| = \|l(\Phi(x))\| = \|\Phi(x)\|_A.$$

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REFERENCES

¹ A. Guichardet, à paraître dans Ann. sci. École norm. supér., ² M. Takesaki, On the Cross-Norm of the Direct Product of C^* -Algebras (in press). ³ T. Tumoraru, Tohoku Math. J., 4, 242 (1952). ⁴ A. Wulfsohn, Bull. Sci. Math., 87, No. 1-2, 13 (1963).

Note: Figure translations are in progress. See original paper for figures.

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