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Abstract

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MASS RELATIONS IN THE OCTET MODEL IN THE PRESENCE OF MIXING

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1. In the octet model ([1]) of particle symmetry, based on the group SU_3, **mass relations** are of special interest. They are derived under the assumption that the mass operator possesses definite transformation properties (is a tensor operator). In ([2]) a new method was proposed for studying tensor operators, which, in particular, is convenient for deriving mass relations. The case of absence of mixing was considered (when each particle belongs to one definite unitary multiplet), and under this assumption an explicit form of the mass relations was obtained for an arbitrary multiplet of particles. Here we consider mixing of an arbitrary (finite) number of multiplets and find explicitly some mass relations valid in this case.

2. Let us denote ([3]) an irreducible representation of the group SU_3 by $\mu = (p, q)$, and a basis vector of this representation by $|\mu\nu\rangle$ or $\Phi(\mu,\nu)$, where $\nu = (y, t, t_3)$ is the triple of quantum numbers (hypercharge, isospin, and isospin projection). An irreducible **tensor operator** is a set of operators which transform into one another as the basis vectors of an irreducible representation.* Accordingly, we shall denote such operators by $T^{\mu}_{\nu} = T^{(p,q)}_{(y,t,t_3)}$. We shall be interested only in the operators $T^{(p)} = T^{(p,p)}_{(0,0,0)}$, and only in the combinations
$M^{(p)} = T^{(0)} + T^{(1)} + ... + T^{(p)}$. In the octet model it is assumed (or derived in first order of perturbation theory) that the mass operator has the transformation properties $M = M^{(1)}$. Then its positive powers are $M^p = M^{(p)}$.

Let the operator T^{μ}_{ν} act in the space spanned by the vectors $|\mu'\nu'\rangle$. Then

$$T^{\mu}_{\nu} = \sum_{\mu_1,\mu_2,\nu_1,\nu_2} |\mu_1\nu_1\rangle\langle\mu_2\nu_2| \cdot \langle\mu_1\nu_1|T^{\mu}_{\nu}|\mu_2\nu_2\rangle. \tag{1}$$

Repeating the tensor structure of this equality, but ignoring its operator structure, one may write

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$$\sum_{\mu_1,\mu_2,\nu_1,\nu_2} \Phi(\mu_1,\nu_1)\,\Phi^*(\mu_2,\nu_2)\langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle = \widetilde{\Phi}(\mu,\nu) \tag{2}$$

(the vector $\widetilde{\Phi}(\mu,\nu)$ is not normalized). That part of the sum (2) which corresponds to fixed definite μ_1, μ_2 transforms independently of the remaining part; therefore

$$\sum_{\nu_1,\nu_2} \Phi(\mu_1,\nu_1)\Phi^*(\mu_2,\nu_2)\langle\mu_1\nu_1|T^\mu_\nu|\mu_2\nu_2\rangle = \Phi^{\mu_1\mu_2}(\mu,\nu). \tag{3}$$

* By transformation of an operator is meant the similarity transformation UTU^{-1}, performed with the aid of an element U of the group.

An analogous equality can also be written for a **reducible** tensor operator, only then on the right-hand side of (3) there will stand a sum of vectors belonging to different μ. In particular, for the operator $M^{(p)}$

$$\sum_{\nu_1,\nu_2} \Phi(\mu_1,\nu_1)\,\Phi^*(\mu_2,\nu_2)\langle\mu_1\nu_1|M^{(p)}|\mu_2\nu_2\rangle = \sum_{s=0}^{p} \Phi^{\mu_1\mu_2}((s,s),(0,0,0)). \tag{4}$$

3. This equality is the starting point for the investigation of the operators $M^{(p)}$. First of all, with its help it is easy to prove that the matrix elements $\langle\mu_1\nu_1|M^{(p)}|\mu_2\nu_2\rangle$ are different from zero only when $\nu_1 = \nu_2 = \nu = (y,t,t_3)$ and do not depend on t_3:

$$\langle\mu_1 ytt_3|M^{(p)}|\mu_2 ytt_3\rangle = m^{(p)}_{\mu_1\mu_2}(y,t). \tag{5}$$

For this it is sufficient to use the rule for addition of the quantum numbers y, t, t_3 (where, for the addition of t, the Clebsch–Gordan coefficients of the subgroup of isotopic rotations SU_2 are to be used).

Further consequences of (4) can be obtained with the aid of the operator K_+ (in the notation of (3)). This is a generator of the group SU_3, which increases y by 1 and t_3 by 1/2. The explicit form of this operator (its action on $\Phi(\mu,\nu)$) is given in (3). Acting on the right-hand side of (4) by the operator $(K_+)^{p+1}$, we obtain zero. Therefore the result of the action on the left-hand side must also be equated to zero. This gives relations between the matrix elements of the operator $M^{(p)}$, which are consequences of its transformation properties. We shall give here the relations between the diagonal elements $m^{(p)}_{\mu\mu}(y,t) = m^{(p)}_\mu(y,t)$. In doing so we shall use terminology from graph theory.

4. Since the quantities $m^{(p)}_\mu(y,t)$ do not depend on t_3, it is convenient to operate with whole isotopic multiplets, and not with their individual components. An isotopic multiplet as a whole is characterized by only two

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quantum numbers (y, t). We shall denote this pair, as before, by the letter ν. This will not lead to misunderstanding. In (4, 5) it is shown which isotopic multiplets ν enter into the basis of the irreducible representation $\mu = (p, q)$. If they are represented by points on a diagram with axes $(2t, y)$, one obtains a two-dimensional lattice with unit spacing along the directions $\pm\frac{1}{2}y + t$ and bounded by the inequalities

$$-\frac{1}{3}(p-q) \le \frac{1}{2}y + t \le \frac{1}{3}(2p+q),$$

$$\frac{1}{3}(p-q) \le -\frac{1}{2}y + t \le \frac{1}{3}(p+2q). \tag{6}$$

We shall call such a diagram the **graph of the representation** μ and denote it by the same letter μ. The graph of an irreducible representation is thus a rectangle with sides inclined at an angle of 45° to the axes $(2t, y)$.

Let some relation connect the quantities $m_\mu^{(p)}(\nu)$ with different $\nu \in \mu$. The set Γ of such points ν entering into the relation will be called the **graph of this relation** and will be depicted on the diagram with axes $(2t, y)$. The graphs of the relations to be discussed, as well as the graphs of the irreducible representations, are rectangles with sides along the directions $\pm\frac{1}{2}y + t$.

From the very definition it follows that the graph Γ of a relation between the elements $m_\mu^{(p)}(\nu)$, $\nu \in \mu$, belongs entirely to the graph μ. In other respects, it turns out, the relation does not depend on μ. We shall therefore denote such a relation by $E_p(\Gamma)$.

Let us define a path L in the graph Γ as a sequence of points in which neighboring points are separated from one another by one step along one of the directions $\pm\frac{1}{2}y + t$. The length of a path will be called the number of links (steps) in

By a straight (shortest) path between two points we mean a path of minimal length containing these points.* We shall encounter only straight upward paths, i.e., such that one can traverse them, at each step moving upward. For them we shall regard the lower point as the beginning and the upper one as the end.

The distance between two points of a graph will be called the length of the straight path between them. The lowest point v_{n} of the graph will be called its beginning, the upper point v_{k} its end, and the distance between them the length of the graph. The graph Γ of the relation $E_p(\Gamma)$ always has length $p+1$. We shall call the index $l(v)$ of a point $v \in \Gamma$ its distance from the beginning of the graph Γ.

5. Let L be some path in the graph. Introduce the notation

$$Q(L) = \prod_{v \in \text{in } L} \frac{1}{2t+1}, \tag{7}$$

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where $v = (y, t)$, and the product is taken over all internal points of the path. If the path L consists of only two points, we shall set $Q(L) = 1$. If L consists of one point $v = (y, t)$, then we shall take $Q(L) = 2t + 1$. Let us also denote

$$S(v_1, v_2) = \sum_{L(v_1, v_2)} Q(L), \tag{8}$$

where the sum is taken over all straight paths between the points v_1 and v_2.

Theorem 1. *For any rectangular graph $\Gamma \subset \mu$ of length $p + 1$, the matrix elements $m_\mu^{(p)}(v)$, $v \in \Gamma$, satisfy the relation*

$$E_p(\Gamma): \quad \sum_{L(v_\mathrm{n}, v_\mathrm{k})} Q(L) \sum_{v \in L} (-1)^{l(v)} C_{p+1}^{l(v)} m_\mu^{(p)}(v) = 0, \tag{9}$$

where the first sum is taken over all straight paths between the beginning v_n and the end v_k of the graph Γ; C_n^m is a binomial coefficient; $l(v)$ is the index of the point v of the graph Γ. Another form of the same relation is

$$E_p(\Gamma): \quad \sum_{v \in \Gamma} (-1)^{l(v)} C_{p+1}^{l(v)} \frac{S(v_\mathrm{n}, v) S(v, v_\mathrm{k})}{2t + 1} m_\mu^{(p)}(v) = 0, \tag{10}$$

where $v = (y, t)$.

Consider the simple special case in which the rectangular graph Γ degenerates into a segment of a straight line. Then between the ends of the graph there exists only one path L. In (9) only one term remains from the first sum. Dividing by $Q(L)$, we obtain

$$E_p(\Gamma),\ \Gamma = L: \quad \sum_{v \in \Gamma} (-1)^{l(v)} C_{p+1}^{l(v)} m_\mu^{(p)}(v) = 0. \tag{11}$$

This relation means that the finite difference of order $(p + 1)$ of the function $m_\mu^{(p)}(y, t)$ along any of the directions $\pm\frac{1}{2}y + t$ is equal to zero. Consequently, this function is a polynomial of degree p in each of the variables $\pm\frac{1}{2}y + t$, and hence of degree $2p$ in each of the variables y and t. This result is obtained as a consequence only of those relations $E_p(\Gamma)$ which have "linear" graphs Γ. The entire set of relations $E_p(\Gamma)$, $\Gamma \subset \mu$, is equivalent to the fact that the function $m_\mu^{(p)}(y, t)$ has the form (5)

$$m_\mu^{(p)}(y, t) = \sum_{i=0}^{p} \sum_{j=0}^{i} a_{ij} y^j \bar{y}^{i-j}, \tag{12}$$

where $\bar{y} = t(t + 1) - \frac{1}{4}y^2$.

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* With this definition, more than one straight path may exist between two points. In that case all these paths have the same length.

6. Let us proceed to the derivation of relations between the **eigenvalues** of the mass operator $M = M^{(1)}$ in the presence of mixing. Suppose that n unitary multiplets $\mu_i,\ i \in N = \{1, \dots, n\}$, are mixed. Then a physical particle is a superposition of basis vectors of irreducible representations $\mu_i,\ i \in N$. The quantum numbers ν of the particle must belong to the region $\mu_N = \bigcup_{i \in N} \mu_i$.

We shall, in all possible ways, select subsets $R \subset N$ in the set of indices N. Let R contain r elements $i, \dots, i_r$. To each $R \subset N$ we assign the graph

$$\mu_{(R)} = \bigcap_{i \in R} \mu_i \setminus \bigcup_{j \in N \setminus R} \mu_j. \tag{13}$$

It is not difficult to see that the totality of all $\mu_{(R)},\ R \subset N$, is a partition of the region μ_N into nonintersecting subregions. Moreover, each point ν of the region $\mu_{(R)}$ belongs to all $\mu_i,\ i \in R$, and only to them.

Therefore, if a physical particle has quantum numbers $\nu \in \mu_{(R)}$, then it is a superposition of the basis vectors $|\mu_i \nu\rangle,\ i \in R$. Particles are eigenvectors of the mass operator M; therefore the latter has matrix elements of the form $\langle \mu_i \nu | M | \mu_j \nu \rangle;\ i, j \in R$. Obviously, the matrix elements with some fixed $\nu \in \mu_{(R)}$ form a separate block. This block $M(\nu)$ is a square matrix of dimension r. Denote its eigenvalues by $M_1(\nu), \dots, M_r(\nu)$. At the same time they are eigenvalues of the entire operator M. Raising them to an arbitrary positive integer power p, we form the quantities $M^{(p)}(\nu) = M_1^p(\nu) + \dots + M_r^p(\nu)$.

It is now not difficult to obtain the main result, using the fact that the form of the relation $E_p(\Gamma)$ does not depend on μ, and applying to the matrix $M(\nu)$ two well-known facts from algebra: 1) the sum of the eigenvalues of a matrix is equal to its trace; 2) if the eigenvalues of a matrix A are $\lambda_1, \dots, \lambda_r$, then the eigenvalues of its power A^p are $\lambda_1^p, \dots, \lambda_r^p$. With the aid of these facts we obtain the following theorem.

Theorem 2. *If $M = M^{(1)}$, then for any region $\mu_{(R)},\ R \subset N$, and any graph $\Gamma \subset \mu_{(R)}$ of length $p+1,\ p \le r$, the quantities $M^{(p)}(\nu)$ satisfy the relations $E_p(\Gamma)$.*

Thus we have obtained relations between the eigenvalues of the mass operator M (i.e., between masses) which are a consequence only of its transformation properties ($M = M^{(1)}$). In fact, in Theorem 2 the restriction $p \le r$ can be removed, which gives additional relations. But these relations are not independent, and follow from those listed in Theorem 2.

Let us note that Theorem 2 does not give **all** independent mass relations. There may exist unaccounted-for relations. In each concrete case some of them can be found by directly using Theorem 1.

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Finally, one last remark. We have investigated the mass operator in first order of perturbation theory ($M = M^{(1)}$). But the results are trivially generalized also to the case of second ($M = M^{(2)}$) and higher orders.

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Note: Figure translations are in progress. See original paper for figures.

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