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Abstract

Full Text

MATHEMATICS

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ON THE METHOD OF CHARACTERISTIC COEFFICIENTS

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1. Consider the class T of continuous distribution functions $F(x)$ possessing the following properties:

a) $0 < F(x_1) < F(x_2) < 1$, if $-\infty < x_1 < x_2 < \infty$; b) $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$.

Let $F(x)$ be one of the functions of the class T; with its aid we form a continuous mapping

$$z = 2\pi(F(x) - 1/2) \tag{1}$$

of the entire real axis onto the interval $[-\pi, \pi]$; it is known that the random variable z will then already be uniformly distributed on this interval.

Let $H(x)$ be an arbitrary distribution function; for it we form a sequence of complex numbers (for $k = 1, 2, ...$)

$$\lambda_F(k, H) = \int_{-\infty}^{\infty} e^{2k\pi i(F(x)-1/2)}\, dH(x) = \alpha_F(k, H) + i\beta_F(k, H). \tag{2}$$

We shall call the numbers (2) the characteristic coefficients (c.c.) of the distribution function $H(x)$ with respect to the distribution function $F(x)$, where

$$\alpha_F(k, H) = \mathbf{M} \cos k2\pi(F(x) - 1/2) = \mathbf{M}T_k(\cos z); \tag{3}$$

$$\beta_F(k, H) = \mathbf{M} \sin k2\pi(F(x) - 1/2) = \mathbf{M}U_k(\cos z), \tag{4}$$

where $T_k(\cos z)$ and $U_k(\cos z)$ are Chebyshev polynomials, respectively of the first and second kind, and z is defined by formula (1).

In particular, if as $F(x)$ one uses the distribution function of the Cauchy law, i.e. sets $F(x) = 1/2 + \operatorname{arctg} x/\pi$, then one obtains the results considered in more detail in ([1]). As in ([1]), with the aid of the generating functions of the Chebyshev polynomials one finds the generating function of the c.c.*

$$\mathfrak{T}_F(u,H)=\int_{-\infty}^{\infty}\frac{1/2(1-u^2)+iu\sin 2\pi(F(x)-1/2)}{1-2u\cos 2\pi(F(x)-1/2)+u^2}\,dH(x);\qquad |u|<1, \quad (5)$$

which we shall call the Chebyshev transform of the distribution function H with respect to the distribution function F.

From (2) and (5), when $H=F$, we immediately derive

$$\lambda_F(k,F)=0,\qquad k=1,2,\ldots; \quad (6)$$

$$\mathfrak{T}_F(u,F)=1/2. \quad (7)$$

Let us note that (6) in this case signifies the general-mean orthogonality of the systems of functions $\{\cos kz\}$ and $\{\sin kz\}$ and 1 on the interval $[-\pi,\pi]$, since in this case z is uniformly distributed on the interval $[-\pi,\pi]$, and the weight in formulas (3) and (4) will be constant.

Let us note that if F and H are both distribution functions of symmetric laws, then $\lambda_F(k,H)$ and $\mathfrak{T}_F(u,H)$ are real and have more

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$$^*\mathfrak{T}_F(u,H)=1/2+\sum_{k=1}^{\infty}\lambda_F(k,H)u^k.$$

simple expressions

$$\lambda_F(k,H)=2\int_0^{\infty}\cos 2k\pi(F(x)-1/2)\,dH(x),\qquad k=1,2,\ldots; \quad (2')$$

$$\mathfrak{L}_F(u,H)=(1-u^2)\int_0^{\infty}\frac{dH(x)}{1-2u\cos 2\pi(F(x)-1/2)+u^2}. \quad (5')$$

2. Let us express the difference $H(x)-F(x)$ in terms of the characteristic coefficients (3), (4). Putting

$$x=F^{-1}((z+\pi)/2\pi), \quad (8)$$

which is possible by virtue of the monotonicity of $F(x)$, we expand the difference $\widetilde{H}(z) - (z+\pi)/2\pi$, where $\widetilde{H}(z) = H[F^{-1}((z+\pi)/2\pi)]$, in a Fourier series on the interval $-\pi \le z \le \pi$; returning to the old variables, we obtain:

$$H(x) = F(x) + 1/2 - \int_{-\infty}^{\infty} F(x)\,dH(x) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\alpha_F(k,H)}{k}\sin 2\pi k(F(x)-1/2) - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\beta_F(k,H)}{k}\cos 2\pi k(F(x)-1/2). \tag{9}$$

If both functions $H(x)$ and $F(x)$ are symmetric, then (9) is simplified:

$$H(x) = F(x) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\lambda_F(k,H)}{k}\sin 2\pi k(F(x)-1/2). \tag{9'}$$

The series (9) and (9) converge to $H(x)$ at all points of continuity of $H(x)$; moreover, they converge in mean, since the integral

$$\rho_F^2(H,F) = \int_{-\infty}^{\infty}[H(x)-F(x)]^2\,dF(x) \tag{10}$$

exists for any distribution functions.

A simple consequence of the expansions given is

Theorem 1. *If the condition*

$$\sum_{k=1}^{\infty}|\lambda_F(k,H)|\,k^{-1} < \infty, \tag{11}$$

is satisfied, then $H(x)$ is continuous, and the series (9) and (9) converge absolutely and uniformly on the entire real axis.

Remark. The expansions (9) and (9) show that, if $F(x)$ is known, then any distribution function $H(x)$ is completely determined by the sequence of characteristic coefficients $\{\lambda_F(k,H)\}$.

By virtue of the orthogonality of the trigonometric functions of multiple arcs, we obtain, respectively, in the general and symmetric cases:

$$\int_{-\infty}^{\infty}[H(x)-F(x)]^2\,dF(x) = \lambda_F^2(0,H) + \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{\alpha_F^2(k,H)+\beta_F^2(k,H)}{k^2}; \tag{12}$$

$$\int_{-\infty}^{\infty} [H(x) - F(x)]^2 \, dF(x) = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{\lambda_F^2(k, H)}{k^2}, \tag{12'}$$

where $\lambda_F(0, H) = 1/2 - \int_{-\infty}^{\infty} F(x) \, dH(x)$. Since

$$\alpha_F^2(k, H) + \beta_F^2(k, H) = |\lambda_F(k, H)|^2 \leq 1, \qquad k = 1, 2, \dots, \tag{13}$$

the series (12) and (12) converge no more slowly than the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Definition. We shall call expression (10) the square of the distance from the distribution function $H(x)$ to the distribution function $F(x)$ of the class T.

If $H(x)$ also belongs to the class T, then one may consider the sequence of characteristic coefficients $\{\lambda_H(k, F)\}$ and the square of the distance $\rho_H^2(F, H)$. But since for any two laws of the class T the easily verified identity

$$\rho^2(H, F) = \int_{-\infty}^{\infty} [H(x) - F(x)]^2 \, dF(x) = \int_{-\infty}^{\infty} [H(x) - F(x)]^2 \, dH(x), \tag{10'}$$

holds, it is natural in this case to call $\rho(H, F)$ the distance between two distribution functions.

It follows from (10) that for two sequences of characteristic coefficients $\{\lambda_F(k, H)\}$ and $\{\lambda_H(k, F)\}$ the following identity is valid:

$$\lambda_F^2(0, H) + \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{|\lambda_F^2(k, H)|}{k^2} = \lambda_H^2(0, F) + \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{|\lambda_H^2(k, F)|}{k^2}, \tag{14}$$

which may be useful in summing series.

3. With the aid of the notions introduced, we shall now formulate the following integral limit theorems.

Consider a sequence of distribution functions $\{F_n(x)\}$ and a putative limiting distribution function $F(x)$, and suppose that the latter belongs to the class T.

Theorem 2. *For convergence in the mean of the sequence $\{F_n(x)\}$ to the distribution function $F(x)$ of the class T, it is necessary and sufficient that the conditions*

$$\lim_{n \to \infty} \lambda_F(k, F_n) = 0, \qquad k = 0, 1, 2, \dots \tag{15}$$

be fulfilled.

Proof follows immediately from the formula

$$\int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \, dF(x) = \lambda_F^2(0, F_n) + \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{|\lambda_F(k, F_n)|^2}{k^2}. \tag{16}$$

Theorem 3. *For convergence in the mean of the sequence of distribution functions* $\{H_n(x)\}$ *to the distribution function* $H(x)$*, it is necessary and sufficient that the conditions*

$$\lim_{n\to\infty} |\lambda_F(k, H_n) - \lambda_F(k, H)| \to 0, \qquad k = 0, 1, 2, \dots, \tag{17}$$

be fulfilled, where F *is some function of the class* T.

Proof follows immediately from the formula

$$\int_{-\infty}^{\infty} [H_n(x) - H(x)]^2 \, dF(x) = [\lambda_F(0, H_n) - \lambda_F(0, H)]^2 +$$

$$+ \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{|\lambda_F(k, H_n) - \lambda_F(k, H)|^2}{k^2}. \tag{18}$$

In conclusion it should be noted that convergence in the mean is equivalent to convergence at points of continuity, and therefore we shall not dwell on this last form of convergence.

4. If the distribution laws $H(x)$ and $F(x)$ have bounded densities $h(x)$ and $p(x)$, and if the latter density is continuous and positive, with $F(x)$ belonging to the class T, then at the points of continuity of $h(x)$

$$\frac{h(x)}{p(x)} = 1 + 2 \sum_{k=1}^{\infty} \{\alpha_F(k, H) \cos 2\pi k(F(x) - 1/2) + \beta_F(k, H) \sin 2\pi k(F(x) - 1/2)\}, \tag{19}$$

and in the symmetric case

$$\frac{h(x)}{p(x)} = 1 + 2 \sum_{k=1}^{\infty} \lambda_F(k, H) \cos 2\pi k(F(x) - 1/2). \tag{19'}$$

Moreover, if

$$\int_{-\infty}^{\infty} \frac{h^2(x)}{p(x)} \, dx < \infty, \tag{20}$$

then

$$\rho_p^2(h,p) = \int_{-\infty}^{\infty} \frac{h^2(x)}{p(x)}\,dx - 1 = 2\sum_{k=1}^{\infty}\{\alpha_F^2(k,h) + \beta_F^2(k,H)\}; \tag{21}$$

this series converges, and $\rho_p^2(h,p)$ may be called the square of the distance from the density h to the density p. In general, condition (20) and the condition of convergence of the series (21) are equivalent.

Finally, if the condition

$$\sum_{k=1}^{\infty} |\lambda_F(k,H)| < \infty, \tag{22}$$

is satisfied, then the law $H(x)$ has a continuous density and the series (19) converges absolutely and uniformly on the entire real axis.

Theorem 4. *For convergence in the mean of a sequence $\{p_n(x)\}$ of densities to the density $p(x) = F'(x) > 0$ (where $F(x)$ belongs to the class T), it is necessary and sufficient that two conditions be fulfilled:*

a) *the series $\sum_{k=1}^{\infty} |\lambda_F(k,F_n)|^2 = \Sigma_n^2$ converges;*

b) *the sum of the series $\Sigma_n^2 \to 0$ as $n \to \infty$.*

Proof follows directly from the formula

$$\int_{-\infty}^{\infty} \frac{p_n^2(x)}{p(x)}\,dx - 1 = \int_{-\infty}^{\infty} \frac{[p_n(x) - p(x)]^2}{p(x)}\,dx = 2\sum_{k=1}^{\infty} |\lambda_F(k,F_n)|^2. \tag{23}$$

5. **Remark 1.** From the convergence of distribution functions under the conditions of Theorem 3 there immediately follows the convergence of the corresponding characteristic functions (uniformly in any finite interval); here it is expedient to use the following expression of the characteristic function in terms of the distribution function:

$$\varphi(t) = 1 + it\left\{\int_0^{\infty} e^{itx}[1 - F(x)]\,dx - \int_{-\infty}^{0} e^{itx}F(x)\,dx\right\}. \tag{24}$$

Remark 2. Theorem 3 can be used directly in practice, since the characteristic coefficients $\lambda_F(k,H)$ may be replaced by their statistical estimates. According to (2), the characteristic coefficients are mathematical expectations of Chebyshev polynomials of known arguments ($F(x)$ is assumed known); for finding numerical values of Chebyshev polynomials one may recommend the tables ([2]).

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CITED LITERATURE

(1) O. V. Sarmanov, *DAN*, **162**, No. 2 (1965).
(2) *Tables of Chebyshev Polynomials* $S_n(x)$ and $C_n(x)$, ed. by K. A. Karpov, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.