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Abstract

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ON THE ALGEBRAIC STRUCTURE OF THE FIELD OF SIEGEL MODULAR FUNCTIONS

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As is known, the field of modular functions of one variable is a field of rational functions. The fields of automorphic functions for principal congruence subgroups modulo $q < 6$ are also rational $(^1)$. In the multidimensional case it is known that the field of automorphic functions associated with any arithmetic discrete group acting in C^n is a field of algebraic functions of n variables $(^{2,3})$. The present work is devoted to a more detailed study of the structure of fields of automorphic functions, in particular to the question of when such fields are fields of rational functions. This question is considered for Siegel modular functions. Let us recall their definition.

Let $\mathfrak{H}_p$ be the Siegel half-plane of rank p, i.e. the set of complex symmetric matrices Z of order p with positive definite imaginary part. This is a homogeneous domain in the space $C^{p(p+1)/2}$. On $\mathfrak{H}_p$ the group $\mathrm{Sp}(p, R)$ acts in the following way. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are square matrices of order p, and $M'IM = I$, where

$$I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}; \quad E = E_p$$

is the identity matrix of order p. Then the domain $\mathfrak{H}_p$ is preserved by the analytic transformations

$$Z \to (AZ + B)(CZ + D)^{-1}.$$

The subgroup $\Gamma_p = \mathrm{Sp}(p, Z)$, consisting of integral matrices, is called the **Siegel modular group**. Meromorphic functions on $\mathfrak{H}_p$ invariant with respect to Γ_p ($f(\gamma z) = f(z)$ for $\gamma \in \Gamma$) are called **Siegel modular functions**. We shall consider the principal congruence subgroups $\Gamma_p(q)$ of the group Γ_p modulo q;

they consist of matrices $\gamma \in \Gamma_p$ congruent to E_{2p} modulo q. The corresponding automorphic functions will be called **Siegel modular functions modulo** q.

A. I. Lapin (^{4}) and Igusa (^{5}) showed that for $p = 2,\ q = 1, 2$ the field of Siegel modular functions is a field of rational functions. The main result of the present article is the following theorem.

Theorem. *The field of Siegel modular functions of order* p *modulo* q*, for* $q \geq 6$*;* $q \geq 4,\ p \geq 3$*;* $q \geq 3,\ p \geq 5$*;* $q = 2,\ p \geq 6$*, is not a field of rational functions. For* $p \geq 13$ *this property is possessed by the field of automorphic functions for any subgroup of finite index of the group* Γ_p *that contains no elements of finite order.*

The proof of the theorem is based on the study of the sequence p_k of plurigenera of the corresponding algebraic variety (p_k is the number of linearly independent holomorphic differentials of degree k: $\varphi(z)(dz)^k = \varphi(z_1, \dots, z_n)(dz_1 \cdots dz_n)^k$. The numbers p_k are birational invariants. For a field of rational functions, $p_k = 0$ for all k. Fields of algebraic functions of n variables for which $p_k \sim ak^n$ are usually called fields of general type. In fact, we shall show that the fields indicated in the theorem are fields of general type.

Let $f(Z)(dZ)^k$ be a holomorphic differential of degree k. Then $f(Z)$ is an automorphic form on $\mathfrak{H}_p$ of weight k:

$$f(\gamma Z)\big(j_\gamma(Z)\big)^k = f(Z), \quad \gamma \in \Gamma,$$

where $j_\gamma(Z) = |CZ + D|^{-(p+1)}$. We shall need the asymptotic behavior of the dimension of the space of automorphic forms of weight k, including vector-valued ones, for large k. Let $\chi(\gamma)$ be a unitary representation of the group Γ_p of dimension m. **Automorphic forms of weight** k **and type** χ are analytic vector-functions taking values in C^m and satisfying the functional equation

$$F(\gamma Z)\big(j_\gamma(Z)\big)^k \chi(\gamma) = F(Z).$$

Denote by $N(p, k, \chi)$ the dimension of the space of these forms for Γ_p.

Lemma. *As* $k \to \infty$ *we have*

$$N(p, k, \chi) \sim A(\Omega_p) m \big((p+1)k/4\pi\big)^{p(p+1)/2},$$

where

$$A(\Omega_p) = 2 \prod_{l=1}^{p} \xi(2l), \qquad \left(\xi(t) = \pi^{-t/2} \Gamma\left(\frac{t}{2}\right) \zeta(t)\right)$$

is the invariant volume of the fundamental domain Ω_p (^{6})*;* $\zeta(t)$ *is the Riemann zeta-function.*

The asymptotic formula stated in the lemma is obtained by means of Selberg's method (^{7}) for computing the dimension of the space of automorphic forms.

An automorphic form does not always generate a holomorphic differential. This is always the case for groups with compact fundamental domain not containing elements of finite order. In this case, from the existence of automorphic forms one can draw a conclusion about the nonrationality of the field of automorphic functions. In the general case one must find additional conditions on the automorphic form, considering a nonsingular model of the algebraic variety associated with the field of automorphic functions.

The first step consists in constructing a compactification Ω^n of the fundamental domain Ω of the group Γ, which is an analytic normal space. The existence of such a compactification has been shown for any arithmetic discrete group Γ of analytic automorphisms of a complex symmetric domain D (^{2,3}). According to the scheme set forth in (^{8}), for this one must adjoin to D the Γ-rational components D_i of the boundary of D, and then factor the resulting space $\mathfrak{M}$, with its natural topology and analytic structure, by Γ, whose action can be defined on Γ. The quotient space $\mathfrak{M}/\Gamma$ will be the desired normal compactification Ω^n of the domain Ω. In this process, for each equivalence class with respect to Γ of components D_i, one fundamental domain $\Omega_i = D_i/\Gamma_i$ is adjoined to Ω, where Γ_i is the induced discrete group on D_i. Recall that the set of components is partially ordered: $D_i \succ D_j$ if D_j lies on the boundary of D_i.

We shall then construct another compactification Ω^r of the fundamental domain Ω, which for all arithmetic groups is an analytic normal space, and for a certain class of groups—in particular, for the congruence subgroups of the Siegel modular group $\Gamma_p(q)$, $q \geq 3$—is a nonsingular complex variety. The space Ω^r is a covering of Ω^n, i.e. there exists an analytic projection

$$\pi : \Omega^r \to \Omega^n,$$

and the inverse image of each point $t \in \Omega^n$ is the direct product of an abelian variety and some number of projective lines; $\pi^{-1}(t)$ for $t \in \Omega$ reduces to a point. In the general case $\pi^{-1}(t)$ is constructed as follows. Let $t \in D_0$, where D_0 is some Γ-rational component of the boundary of the domain D. Realize D as a Siegel domain of the third kind with base D_0 (^{2,8}):

$$\operatorname{Im} z - \operatorname{Re} L_t(u, u) \in V, \quad t \in D_0, \quad z \in C^j,$$

$$u \in C^m, \quad V \text{ is a certain cone in } R^j.$$

With each point $t \in D_0$ there is associated a discrete group $\Gamma^{(t)}$ of translations of the space C^m. Then

$$\pi^{-1}(t) = \mathfrak{A}^{(t)} \times (P^1)^s,$$

where $\mathfrak{A}^{(t)} = C^m/\Gamma^{(t)}$, and P^1 is the projective line. We shall not give the formulas in the general case. We shall call the discrete group Γ **regular** if, for any Γ-rational components $D_1 \prec D_2$, there exists-

there are Γ-rational components $D_i,\ D_1 < D_i < D_2$, whose ranks (as symmetric spaces) take all values between the ranks of D_1 and D_2. This property is possessed, in particular, by subgroups of the Siegel modular group. For regular groups Γ, the number of projective lines for the component D_0 is equal to $j - l$, where l is the rank of the cone V.

Introduce a topology in Ω^r. Let (T_i, v_i) be a sequence of points of Ω^r, $T_i \in \Omega^n$, $v_i \in \pi^{-1}(T_i)$. One may restrict oneself to the case when the T_i belong to one component D_0. Let (T, v) be a point of Ω^r, and let T belong to the component $D_1 < D_0$. Then D_0 can be represented as a Siegel domain of the third kind with base D_1: $T_i = (z_i, u_i, t_i)$, $t_i \in D_1$. We set: $(T_i, v_i) \to (T, v)$, if $T_i \to T$ in the topology of Ω^n, in particular $t_i \to T$, and, in addition, $(u_i, v_i) \to v$. The space Ω^r thus constructed, for regular groups, can have only singularities connected with elements of finite order in Γ. In the general case, apart from these singularities, there can be only singularities analogous to those possessed by the fundamental domain of the Hilbert modular group.

Let now D be the Siegel upper half-plane $\mathfrak{H}_p$; Γ the Siegel modular group Γ_p. It is known that all Γ_p-rational components of the same rank are equivalent to one another with respect to Γ_p and analytically equivalent to the Siegel upper half-plane of the corresponding rank $\mathfrak{H}_{p-1}, \mathfrak{H}_{p-2}, \dots, \mathfrak{H}_1, \mathfrak{H}_0$ ($\mathfrak{H}_0$ is a point); the groups Γ^i then correspond to Siegel modular groups. Thus,

$$\Omega_p^r = \Omega_p + \dots + \Omega_1 + \Omega_0.$$

With the component $D_\nu = \mathfrak{H}_{p-\nu}$ is associated the realization of D as a Siegel domain of the third kind

$$Z = \begin{pmatrix} z & v \\ v' & T \end{pmatrix},$$

where $z \in C^j,\ j = \nu(\nu+1)/2,\ v \in C^m,\ m = (p-\nu)\nu$;

$$T \in \mathfrak{H}_{p-\nu}; \quad [z - i(\operatorname{Im} v)Y^{-1}(\operatorname{Im} v')] \in \mathfrak{H}_\nu.$$

Let

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix};$$

$\Gamma^{(T)}$ consists of the transformations $v \to v + a + \beta T$, where a, β are rectangular matrices of order $\nu \times (p-\nu)$, consisting of complex integers. Then

$$\pi^{-1}(T) = \mathfrak{A}^{(T)} \times (P^1)^s, \qquad \mathfrak{A}^{(T)} = C^m/\Gamma^{(T)}, \quad s = \nu(\nu-1)/2.$$

Let us find the condition on an automorphic form $\varphi(Z)$, generating a holomorphic differential, in a neighborhood of points (T, v) lying over Ω_{p-1}. Expand $\varphi(Z)$ in a Fourier series with respect to the variable $z \in C^1$ (in Γ_p there are transformations $z \to z+n$, n integers):

$$\varphi(z, v, T) = \sum_{r \geq 0} \varphi_r(v, T) e(r, z),$$

where $e(z) = \exp(2\pi i z)$. These series are called Fourier–Jacobi series, since for fixed T the functions $\varphi_r(v, T)$ are Jacobi theta-functions:

$$\varphi_r(v, T) = \varphi_r(v + \beta T, T) e(2r\beta v' + r\beta T \beta'),$$

where β is a row of $(p-1)$ complex integers (one must consider the automorphisms of $\mathfrak{H}_p$: $T \to T$, $v \to v + \beta T$, $z \to z + 2\beta v' + \beta T \beta'$, and use the functional equation for automorphic forms).

In a neighborhood of the points (v, T) the local parameters will be $(v, T, e(z))$. Hence it follows that, in order for the differential $\varphi(Z)(dZ)^k$ to be holomorphic at these points, it is necessary that the functions $\varphi_r(v, T)$ vanish identically for $r < k$. Expand $\varphi_r(v, T)$ in the basis theta-functions:

$$\vartheta_\mu(v, T) = \sum_{m \equiv \mu (2r)} e\left(\frac{mTm'}{4r} + mv'\right),$$

where m, μ are integer rows of length $p-1$. We have

$$\varphi_r(v, T) = \sum_\mu \alpha_\mu^r(T) \vartheta_\mu(v, T),$$

where the summation is over all vectors μ for which $0 \leq \mu_j < 2r$. From the functional equation for automorphic forms it follows that

$$\alpha_\mu^r(T + S) = \alpha_\mu^r(T) e(\mu S \mu'/4r),$$

where S is an integral symmetric matrix;

$$\alpha_{\mu A}^r(T) = \alpha_\mu^r(A' T A),$$

where A is an integral matrix of order $(p-1)$ with determinant 1, and, finally,

$$\alpha_\mu^r(-T^{-1}) = |T|^{(p+1)k} \frac{1}{\sqrt{|1 - iT|}} (2r)^{-(p-1)/2} \times$$

$$\times \sum_\nu{}' a_\mu^r(T) e\left(\frac{\nu\mu'}{2r}\right).$$

It follows from this that $\{a_\mu^r(T)\}$, for fixed r, is a vector-valued automorphic form on $\mathcal{L}_{p-1}$ of weight $k((p+1)/p) - 1/2p$, whose type is a certain representation χ_r of dimension $(2r)^{p-1}$.

Thus, for holomorphy of the differential at the points under consideration, it is sufficient that the dimension of the space of scalar automorphic forms of weight k on $\mathcal{L}_p$ exceed the sum of the dimensions of the spaces of automorphic forms of weight $k((p+1)/p) - 1/2p$ and of types χ_r for $r < k$. In doing so one may restrict oneself to arbitrarily large k, and therefore it is enough to know only the asymptotics of the dimension. The calculation shows that this condition is satisfied for $p \geq 13$. An analogous consideration in a neighborhood of the points (T, v) shows that no new conditions on $\varphi(Z)$ arise (here it is essential that $\varphi(Z)$ be invariant under simultaneous permutations of the rows and the corresponding columns of the matrix Z). It follows that there cannot be rational fields associated with subgroups of finite index of the Siegel group Γ_p, $p \geq 13$, which contain no elements of finite order. Among these subgroups are contained, in particular, the principal congruence subgroups modulo $q \geq 3$. However, in this case one can obtain a sharper estimate by directly finding the number of components and the number of conditions on differentials for these components (see the statement of the theorem).

The principal congruence subgroup modulo 2 ($\Gamma_p(2)$) has elliptic elements only of the second order. Consider a certain maximal set of fixed points $\mathfrak{M}$ and the subgroup $\Gamma(\mathfrak{M})$ of the group $\Gamma_p(2)$ consisting of transformations that preserve every point of $\mathfrak{M}$. It is easy to see that $\Gamma(\mathfrak{M})$ is an abelian group and, since it consists of elements of second order, all elements of $\Gamma(\mathfrak{M})$ are simultaneously reducible to diagonal form, with ± 1 on the diagonal. Then, after making the corresponding transformation, one may assume that $\mathfrak{M}$ is given by the linear conditions $z_1 = 0, \dots, z_\nu = 0$. From the maximality condition of $\mathfrak{M}$ it follows that $\Gamma(\mathfrak{M})$, besides the identity transformation, contains only the transformation that multiplies $z_1, \dots, z_\nu$ by -1, while the remaining coordinates are preserved. It is important that for $\Gamma(\mathfrak{M})$, when $p > 2$, one always has $\nu > 1$. To resolve the singularities it is sufficient, by a σ-process along the entire fixed subvariety, to glue in a $(p-1)$-dimensional projective space. It is easy to verify that the application of σ-processes at the intersection points of different $\mathfrak{M}$' s is compatible. After factorization one obtains a nonsingular variety. Moreover, if $\nu > 1$ (as will be the case for $p > 2$), no new conditions on $\varphi(z)$ arise. As a result we obtain the assertion of the theorem for $\Gamma_p(2)$.

The methods used in this paper are applicable to arbitrary arithmetic discrete groups. Apparently, in this way one can describe all such arithmetic groups for which the fields of automorphic functions are rational.

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Note: Figure translations are in progress. See original paper for figures.