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Abstract

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On W. Blaschke' s Problem on Quasi-Translation Surfaces

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The Blaschke-Grünwald kinematic mapping (^{1}) establishes a one-to-one correspondence between the motions of the Euclidean plane and the points of three-dimensional quasi-elliptic space—a space with an absolute consisting of a pair of complex-conjugate planes and a pair of complex-conjugate points situated on the line of their intersection.

Blaschke calls a **quasi-translation surface** a surface

$$\mathfrak{X} = \mathfrak{A}(u)\mathfrak{B}(v), \tag{1}$$

where $\mathfrak{X}, \mathfrak{A}, \mathfrak{B}$ are quaternions of the form

$$\mathfrak{A} = a_0 e_0 + a_1 e_1 + \varepsilon(a_2 e_2 + a_3 e_3), \qquad \varepsilon^2 = 0,$$

normalized so that

$$a_0^2 + a_1^2 = 1.$$

The choice of the parameters u, v is determined by the conditions

$$a_0(u) = \cos u, \qquad a_1(u) = \sin u, \qquad b_0(v) = \cos v, \qquad b_1(v) = \sin v.$$

Blaschke' s problem consists in finding surfaces that can be represented in the form (1) in two or more ways. In kinematics on the plane this corresponds to finding two-parameter motions representable in two essentially different ways in the form of a product of two one-parameter motions.

In the present note surfaces carrying an infinite set of quasi-translation nets are obtained. It turned out that these are surfaces of revolution of a special kind. Also found are all surfaces of revolution of quasi-elliptic space that are at the same time quasi-translation surfaces. It is proved that they carry four quasi-translation nets. The corresponding two-parameter motions of the Euclidean plane and their decompositions into products of one-parameter motions are determined.

As N. G. Chebotarev proved (^{2}), generalized translation surfaces may have either a continuum of systems of imprimitivity or no more than four. However, it remained unknown whether there exist generalized translation surfaces having all four systems of imprimitivity. The surfaces we have found, carrying four quasi-translation nets, give a positive answer to the question posed by N. G. Chebotarev (^{3}).

1. Let t be a tangent to a surface S at a point O. About S we circumscribe a ruled surface whose generators are right (left) parallels to t. The tangent t' to the line of contact of S with the circumscribed ruled surface will be called **right (left) quasi-conjugate** to t. Analytically, the condition of quasi-conjugacy is written as follows:

$$(b_{ik} + \varepsilon_{ik})\, du^i \delta u^k = 0, \tag{2}$$

where b_{ik} is the second tensor of the surface S,

$$\varepsilon_{ik} = \left(X,\ x,\ \frac{\partial x}{\partial u^i},\ \frac{\partial x}{\partial u^k} \right),$$

X is the pole of the tangent plane to S at the point O with respect to the absolute.

The derivation formulas for the surface S have the form

$$\frac{\partial^2 x}{\partial u^i \partial u^k} = G^j_{ik} \frac{\partial x}{\partial u^j} - \varepsilon_{ik} x + b_{ik} X.$$

We shall call a net of curves a **Chebyshev** net if the tangents to the curves of one family along the curves of the second family are parallel in the sense of Levi-Civita with respect to the connection G^j_{ik}. For a Chebyshev net (v^1, v^2), the Servant–Bianchi equations hold:

$$\frac{\partial^2 u^k}{\partial v^1 \partial v^2} + G^k_{ij} \frac{\partial u^i}{\partial v^1} \frac{\partial u^j}{\partial v^2} = 0. \tag{3}$$

Theorem 1. *In order that a net of curves on a surface be a net of quasi-translation, it is necessary and sufficient that it be Chebyshev and quasi-conjugate.*

2. Suppose that on the surface (1) there exists a net of quasi-translation

$$\frac{dv}{du} = \varphi(u, v), \qquad \frac{\delta v}{\delta u} = \psi(u, v), \tag{4}$$

different from the coordinate net. The conditions (2), (3) for this net take the form:

$$b - 2b'\varphi + b''\varphi\psi = 0; \tag{2’}$$

$$2b'(\varphi_u + \psi\varphi_v) = (\varphi + 1)(b'_u - b'_v\varphi\psi),$$

$$2b'(\psi_u + \varphi\psi_v) = (\psi + 1)(b'_u - b'_v\varphi\psi), \tag{3’}$$

where $b = b_{11}b_{12}$, $b' = b_{12}^2$, $b'' = b_{22}b_{12}$, $\varepsilon_{12} = b_{12}$.

Rewrite $(2')$ as

$$b''\psi - b' = b' - \frac{b}{\varphi} = \sigma.$$

Now from $(3')$ it follows that

$$(\sigma^2 + bb'' - {b'}^2)\sigma_u = \alpha_1\sigma^3 + \beta_1\sigma^2 + \gamma_1\sigma + \delta_1,$$

$$(\sigma^2 + bb'' - {b'}^2)\sigma_v = \alpha^2\sigma^3 + \beta_2\sigma^2 + \gamma_2\sigma + \delta_2, \tag{5}$$

where the coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ are functions of b, b', b'' and their derivatives. The integrability condition of system (5) has the form

$$A\sigma^3 + B\sigma^2 + C\sigma + D = 0, \tag{6}$$

where A, B, C, D are expressed through b, b', b'' and their derivatives.

Suppose the surface carries a continuum of nets of quasi-translation. Then equation (6) is satisfied identically with respect to σ, and system (5) is completely integrable. Equating the coefficients A, B, C, D to zero, we obtain a system of functional equations with respect to the functions $a_2(u), a_3(u), b_2(v), b_3(v)$ of equation (1).

The general solution of this system is:

$$\begin{aligned}
a_2(u) &= a\sin(u+\beta) + \sqrt{\gamma + 2\delta u}\,\cos(u+\vartheta),\\
a_3(u) &= a\cos(u+\beta) + \sqrt{\gamma + 2\delta u}\,\sin(u+\vartheta),\\
b_2(v) &= \bar{a}\sin(v+\bar{\beta}) + \sqrt{\bar{\gamma} + 2\delta v}\,\sin(v-\vartheta),\\
b_3(v) &= \bar{a}\cos(v+\bar{\beta}) + \sqrt{\bar{\gamma} + 2\delta v}\,\cos(v-\vartheta).
\end{aligned}$$

Substituting these values into equation (1), we obtain the canonical equation of the sought surfaces of quasitranslation. Subjecting these surfaces to a motion, one can make the constants a, α vanish.

Introduce new curvilinear coordinates ξ, η:

$$\xi = u + v, \quad \eta = v - u - \vartheta + \operatorname{arc\,tg} \sqrt{\frac{\bar{\gamma} + 2\delta v}{\gamma + 2\delta u}}.$$

The equations of the surface (1) then take the form

$$x_0 = \cos\xi, \quad x_1 = \sin\xi, \quad x_2 = p(\xi)\cos\eta, \quad x_3 = p(\xi)\sin\eta, \tag{7}$$

where

$$p^2(\xi) = \gamma + \bar{\gamma} + 2\delta\xi. \tag{8}$$

This is a surface of revolution. The constants $\gamma, \bar{\gamma}, \delta$ determine the meridian of the surface. To different values of the parameter ϑ there correspond different nets of quasitranslation.

Theorem 2. *The only surfaces carrying a continuum of nets of quasitranslation are the surfaces of revolution (7), (8).*

3. The question naturally arises of finding all surfaces of revolution carrying a finite number of nets of quasitranslation.

Let on the surface of revolution (7) of the quasi-elliptic space there exist a net of quasitranslation

$$\frac{d\eta}{d\xi} = \varphi(\xi, \eta), \quad \frac{\delta\eta}{\delta\xi} = \psi(\xi, \eta).$$

The conditions (2), (3) for it take the form:

$$p(\varphi + 1)(\psi - 1) = p'', \tag{2"}$$

$$\varphi_\xi + \psi\varphi_\eta = -\frac{p'}{p}(\varphi + \psi), \quad \psi_\xi + \varphi\psi_\eta = -\frac{p'}{p}(\varphi + \psi). \tag{3"}$$

The compatibility condition is the vanishing of a polynomial in φ with coefficients depending on ξ; consequently, $\varphi_\eta = \psi_\eta = 0$. Now from equation $(3'')$ it follows that

$$\varphi = \frac{M}{p^2} + N, \quad \psi = \frac{M}{p^2} - N, \tag{9}$$

where M, N are constants, and from $(2'')$, for $N \neq 1$,

$$p^2(\xi) = \frac{\sqrt{H^2 - 4M^2(N-1)^2}}{2(N-1)^2} \sin\left[2|N-1|(\xi + H')\right] + \frac{H}{2(N-1)^2}, \tag{10}$$

where H, H' are constants. In the case $N = 1$,

$$H^2 p^2(\xi) = (H^2 \xi + H')^2 + M^2. \tag{11}$$

Assigning the surface to the net of quasitranslation u, v, we obtain for the quaternions $\mathfrak{A}, \mathfrak{B}$ of equation (1) the expressions

$$\mathfrak{A}(u) = e_0 \cos u + e_1 \sin u + \varepsilon n[\sin(2N-1)(u+H')\, e_2 + \cos(2N-1)(u+H')\, e_3],$$

$$\mathfrak{B}(v) = e_0 \cos v + e_1 \sin v + \varepsilon m[\cos(2N-1)(v+H')\, e_2 + \sin(2N-1)(v+H')\, e_3], \tag{12}$$

where $n = \sqrt{H+2M}/2|N-1|$, $\quad m = \sqrt{H-2M}/2|N-1|$, $\quad N \neq 1$, and for $N = 1$

$$\mathfrak{A}(u) = e_0 \cos u + e_1 \sin u + \varepsilon[-Hu \sin u\, e_2 + Hu \cos u\, e_3],$$

$$\begin{aligned} L(v) = e_0 \cos v + e_1 \sin v + \varepsilon \left\{ \left[\frac{M}{H} \cos v + \left(Hv + \frac{H'}{H} \right) \sin v \right] e_2 + \right. \\ \left. + \left[-\frac{M}{H} \sin v + \left(Hv + \frac{H'}{H} \right) \cos v \right] e_3 \right\}. \end{aligned} \tag{13}$$

When M is replaced by $-M$, or $N-1$ by $1-N$, the meridian $p(\xi)$ of the surface does not change, whereas the nets of quasitranslation change essentially. Consequently, the surfaces found carry four nets of quasitranslation.

Theorem 3. *All surfaces of revolution that are at the same time surfaces of quasitranslation with a finite number of nets are determined by equations (12), (13). They carry four nets of quasitranslation.*

4. By virtue of the Blaschke–Grünwald kinematic mapping (¹), the surfaces (1) correspond to two-parameter motions of the Euclidean plane which admit a continuum of ways of being decomposed into the product of two one-parameter motions.

The first motion can be realized by rolling the equilateral hyperbola

$$(x' \sin\vartheta + y' \cos\vartheta)(x' \cos\vartheta - y' \sin\vartheta) + 2\delta = 0 \tag{14}$$

along the curve

$$\rho^2 = \frac{\delta^2}{\gamma + \delta(\varphi + \vartheta - \pi/2)}; \tag{15}$$

the second motion is realized by rolling the curve

$$\rho^2 = -\frac{\delta}{\varphi + \vartheta}$$

along the hyperbola congruent to the hyperbola (14). The parameter ϑ corresponds to the different ways of decomposition.

The two-parameter motions corresponding to the surfaces (12) are decomposed into the product of two one-parameter motions, which can be realized by rolling a circle of radius $2|n(N-1)|$ along a circle of radius $2|nN|$, and a circle of radius $2|mN|$ along a circle of radius $2|m(N-1)|$. The remaining three ways of decomposition are obtained from this one by replacing M by $-M$ or $N-1$ by $1-N$. For the surfaces (13), one of the centroids degenerates into a straight line.

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Note: Figure translations are in progress. See original paper for figures.

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