ON THE STABILITY OF A RAREFIED PLASMA IN SHORT TRAPS

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Abstract

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ON THE STABILITY OF A RAREFIED PLASMA IN SHORT TRAPS

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1. In studying plasma oscillations in traps of finite length, the equations obtained for an infinitely long system can, generally speaking, be used only in the case when the trap length $2b$ along the magnetic field is sufficiently large, so that the frequency of electron circulation along the trap is much smaller than the oscillation frequency ω: $\omega \gg v_{Te}/b$ (v_{Te} is the thermal velocity of the electrons). In the present work we consider the drift and flute instabilities of a rarefied ($\beta \ll 1$) inhomogeneous plasma in a trap that is "long" with respect to the ions, but "short" with respect to the electrons: $v_{Ti}/b \ll \omega \ll v_{Te}/b$. In this case the electron is acted upon by the perturbing field averaged over the length of the trap, and the equations for the oscillations differ substantially from those that occur for a system of infinite length. We shall restrict ourselves to an idealized problem in which the magnetic lines of force are assumed to be straight along the entire length of the trap, and the latter is bounded at its ends by walls that ideally reflect particles. In what follows (except for Sec. 3), for simplicity we shall consider perturbations of high modes $n \gg 1$ (n is the azimuthal quantum number), for which "plane" (rather than cylindrical) equations may be used.

Let, in the unperturbed quasi-Maxwellian distribution functions [1], the density of the Larmor centers depend on the coordinate x, and let the z-axis be directed along the external magnetic field. Let the perturbation of the potential have the form $\psi = \exp(ik_y y + i\omega t)\varphi(x, z)$. Then, for long ($\partial \ln\psi/\partial z \sim 1/b$) waves, the ion charge density ρ_i, entering Poisson' s equation

$$\Delta\psi = -4\pi(\rho_i + \rho_e), \tag{1}$$

has the same form as in the case of a plasma unbounded in z, and, neglecting the exponentially small anti-Hermitian part, is equal to [2]

$$\rho_i = \hat{\rho}_i\psi = \exp(ik_y y + i\omega t)\frac{e^2 N_0}{T_\perp}\int_{-\infty}^{\infty}\frac{e^{ik_{xx}}\,dk_x}{\sqrt{2\pi}}\times$$
$$\times\int_{-\infty}^{\infty}\frac{q_i(k_x-l)}{\sqrt{2\pi}}\left(2\left\{\left[1+k_y(k_x-l)R_i^2\frac{\omega_i}{2\omega}\right]\left[1-\frac{T_\parallel}{m_i\omega^2}\frac{\partial^2}{\partial z^2}\right]-\frac{T_\perp-T_\parallel}{m_i\omega^2}\frac{\partial^2}{\partial z^2}\right\}\right.$$
$$\left.\times\int_0^\infty te^{-t^2}J_0\left(\sqrt{k_x^2+k_y^2}\,R_i t\right)J_0\left(\sqrt{l^2+k_y^2}\,R_i t\right)\,dt-1\right)\varphi(l,z)\,dl, \tag{2}$$

where N_0 is a quantity of the dimension of density; $\varphi(l,z)$ is the Fourier transform of the function $\varphi(x,z)$; q_i is the Fourier transform of the density of ion Larmor centers; m_i, ω_i, R_i, $T_\parallel$ and $T_\perp$ are, respectively, the mass, gyrofrequency, mean Larmor radius, longitudinal and transverse temperature.

ions. The electron charge density ρ_e, for $|\operatorname{Re}\omega| \gg |\operatorname{Im}\omega|$, in the limiting case of interest to us, $\omega b\sqrt{m_e/T} \ll 1$, has the form

$$\rho_e = \hat{\rho}_e\psi = \exp(ik_y y + i\omega t)\,\frac{e^2 N_0}{T}\left\{\left[\omega Q_e - \frac{k_y}{2}R_e^2|\omega_e|\frac{dQ_e}{dx}\right]\times\right.$$
$$\times\left[\frac{1}{2b\omega}\int_{-b}^{b}\varphi(x,z')\,dz' + i\sqrt{\frac{2m_e}{\pi T}}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-b}^{b}\cos\frac{\pi n}{2b}(z+b)\times\right. \tag{3}$$
$$\left.\left.\times\cos\frac{\pi n}{2b}(z'+b)\,\varphi(x,z')\,dz'\right]-Q_e\varphi(x,z)\right\},$$

where Q_e is the density of the electron Larmor centers; T, ω_e, R_e are, respectively, the temperature, gyrofrequency, and Larmor radius of the electrons.

The main difference between equation (1), with ρ_i and ρ_e determined by formulas (2), (3), and the corresponding equation for the case of a plasma unbounded in z, consists in the presence in (1) of the integral term

$$\frac{2\pi e^2 N_0}{Tb\omega}\left(\omega Q_e - \frac{k_y}{2}R_e^2|\omega_e|\frac{dQ_e}{dx}\right)\int_{-b}^{b}\varphi(x,z)\,dz.$$

Let us note that this term is by no means "literally" small. Therefore the spectrum of oscillations determined by (1), generally speaking, differs substantially from the spectrum of an infinitely long system.

Assume that the boundary conditions at the ends $z = \pm b$ have the form $h_1\partial\varphi/\partial z \pm h_2\varphi = 0$, $h_1 \geq 0$, $h_2 > 0$. The condition on the lateral surface is immaterial for what follows, since we are interested in quasiclassical solutions with wavelength in the transverse direction much smaller than the cylinder radius a: $k_\perp a \gg 1$.

2. Multiplying (1) by the complex-conjugate function ψ^* and integrating over the volume occupied by the plasma, one can obtain an algebraic equation which the frequency ω must necessarily satisfy. If the small (of order $\omega b\sqrt{m_e/T}$) anti-Hermitian part of $\hat{\rho}_e$ is neglected, this equation has no complex roots, so that hydrodynamic instability is absent in the system. The condition for kinetic drift instability of the system under consideration (oscillations can be amplified by resonant electrons whose bounce frequency along the trap, or integer multiples of it, coincides with the wave frequency) for the quasiclassical solutions of interest to us ($k_\perp a \gg 1$) coincides with the corresponding condition for an infinite system: $\omega < \frac{1}{2} k_{yR} e^2 |\omega_e| \, d \ln Q_e/dx$.

3. We shall begin the clarification of the question of the influence of the integral term in (1) with the case $T_\perp = T_\parallel$. In this case the term in (2) associated with pressure anisotropy vanishes. To shorten the notation, we shall set $T_\perp = T$. In order to simplify the discussion, we first turn to the case of a very rarefied plasma, when $k_\perp^2 d^2 \gg 1$, $d = (T/4\pi e^2 N_0)^{1/2}$ is the Debye radius. Then, in the right-hand side of (1), one must retain only the terms connected with inhomogeneity and containing the derivative of the density. Let us consider the case of cylindrical geometry, when the perturbation has the form $\psi \sim e^{in\varphi + i\omega t}\psi(r,z)$. In order that the oscillation equation be as simple as possible—with constant coefficients—we assume that the plasma density in a cylinder of radius a with a metallic wall decreases according to the law $N(r) = N_0(1 - r^2/a^2)$, $b \gg a \gg R_i$.

Then the equation of oscillations will be $\left(\Delta_\perp = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{n^2}{r^2}\right)$:

$$\Delta_\perp \psi + \frac{\partial^2 \psi}{\partial z^2} = -\frac{8\pi e^2 N_0 n}{m_i a^2 \omega_i \omega}\left\{\frac{1}{2b}\int_{-b}^{b} \psi(r,z')\,dz' - 2\int_0^\infty t e^{-t^2} J_0^2\left(\sqrt{-\Delta_\perp}\, R_i t\right) dt \cdot \left[\psi - \frac{T}{m_i \omega^2}\frac{\partial^2 \psi}{\partial z^2}\right]\right\}. \tag{4}$$

We shall seek the solution of (4) in the form $\psi = J_n(\alpha_k r/a)\varphi(z)$, where α_k are the roots of the Bessel function J_n. Let us denote the integral independent of z

$$\frac{1}{2b}\int_{-b}^{b} \varphi(z)\,dz$$

by c. Then the solution satisfying the boundary conditions at $z = \pm b$ will be

$$\varphi(z) = \frac{2\omega_{0i}^2 n}{2\omega_{0i}^2 n A + \alpha_k^2 \omega_i \omega}\left(1 + \frac{h_2 \cos\gamma z}{h_1 \gamma \sin\gamma b - h_2 \cos\gamma b}\right) c, \tag{5}$$

where

$$\omega_{0i}^2 = \frac{4\pi e^2 N_0}{m_i}, \quad A = \exp\left(-\frac{\alpha_k^2 R_i^2}{2a^2}\right) I_0\left(\frac{\alpha_k^2 R_i^2}{2a^2}\right), \quad \gamma = \left(-\frac{\alpha_k^2 \omega_i \omega + 2\omega_{0i}^2 nA}{a^2\omega_i\omega + 2\omega_{0i}^2 nAT/m_i\omega^2}\right)^{1/2}.$$

The integral of (5) with respect to z from $-b$ to b must be equal to $2bc$. Hence we obtain the dispersion equation for the frequency ω:

$$\frac{2\omega_{0i}^2 n}{2\omega_{0i}^2 nA + \alpha_k^2\omega_i\omega}\left[1 + \frac{h_2 \sin\gamma b}{\gamma b(h_1\gamma\sin\gamma b - h_2\cos\gamma b)}\right] = 1. \tag{6}$$

It is easy to see that, because of the smallness of the parameter $T/m_i\omega^2 b^2 = \varepsilon$ (i.e., the smallness of the coefficient multiplying $\partial^2\psi/\partial z^2$ in (4)), the roots of (6), to terms of order ε, are equal to the roots of the dispersion equation for an infinite cylinder,

$$2\omega_{0i}^2 nA + \alpha_k^2\omega_i\omega = 0,$$

which is obtained if the integral term in (4) is discarded. Although this term is not *a priori* small, the actual solution is such that the integral

$$\int_{-b}^{b} \psi(x,z)\, dz$$

vanishes (to accuracy ε). The smallness of the coefficient multiplying $\partial^2\psi/\partial z^2$ in (1) is inherent in the initial approximation in general and is not specifically connected with the particular case considered (solely for maximum simplification) $k_\perp^2 d^2 \gg 1$, $\frac{1}{r}\frac{dN}{dr} = \text{const}$. Therefore the conclusion that the integral electron term does not affect the oscillation frequency and, consequently, that the drift instability occurs also in a "short" trap extends to all cases in which $T_\parallel = T_\perp$ (in fact, it is sufficient that $T_\parallel \gtrsim T_\perp$).

4. Let us now consider the case $T_\parallel \ll T_\perp$, when, on the one hand, the quantity $\sqrt{T_\parallel/m_i}\,\partial\ln\psi/\partial z$ is small compared with the frequency ω, so that, with respect to the ions, the trap is long, while, on the other hand, the coefficient multiplying $\partial^2\psi/\partial z^2$ in (1) may be (owing to the term associated with the pressure anisotropy) not small. Since we restrict ourselves to the case of weakly inhomogeneous plasma, $R_i \ll a$, for quasiclassical (in the direction across the magnetic field) solutions $k_\perp a \gg 1$ one may, as is usually done, use simplified equations in which the dependence of the equilibrium quantities $N(r)$ and $\frac{1}{r}\frac{dN}{dr}$ on the coordinate r (or, in the planar case, on x) is neglected. For definiteness we shall assume that the boundary conditions at $z = \pm b$ have the form $\psi_{z=\pm b} = 0$. By the method described in Sec. 3, we obtain the dispersion equation $\left(\Omega = \frac{1}{2}k_{yR}i^2\omega_i \times d\ln N/dx\right.$ is the drift frequency, $A = \exp(-\frac{1}{2}k_\perp^2 R_i^2) I_0(\frac{1}{2}k_\perp^2 R_i^2)$):

$$1-\frac{\tan\left[\frac{m_i\omega^2}{T_\perp A}\left(1+\frac{T_\perp}{T}-A-A\frac{\Omega}{\omega}\right)\right]^{1/2}b}{\left[\frac{m_i\omega^2}{T_\perp A}\left(1+\frac{T_\perp}{T}-A-A\frac{\Omega}{\omega}\right)\right]^{1/2}b}=\frac{1+\frac{T_\perp}{T}-A-A\frac{\Omega}{\omega}}{\frac{T_\perp}{T}-\frac{\Omega}{\omega}}. \tag{7}$$

Equation (7) has unstable $\left(\omega<\dfrac{T}{T_\perp}\Omega\right)$ solutions if

$$A<\frac{(T_\perp+T)m_i\Omega^2b^2}{(T_\perp+T)m_i\Omega^2b^2+\pi^2T_\perp^3/4T}. \tag{8}$$

Exactly the same condition is obtained if, using the equations for an infinite system, the finiteness of the trap length is taken into account only by quantizing the wave number k_z ($k_z=\pi/2b$). This circumstance becomes clear if one notes that, at the instability boundary $\omega=\dfrac{T}{T_\perp}\Omega$, the coefficient of the integral term in (1) vanishes.

Thus, also for $T_\| \ll T_\perp$, the criterion for the drift instability of the finite system considered coincides with the criterion obtained for a system of infinite length (although the oscillation spectra themselves are substantially different for the two systems).

5. In conclusion let us consider the flute instability in a short trap. Its cause may be the curvature of the magnetic-field lines, which causes the drift of particles in the azimuthal direction. Let the drift velocity of the electrons relative to the ions be v_0. Then, neglecting the anti-Hermitian part in $\hat{\rho}_e$, we obtain (for $T_\| \ll T_\perp$) the following equation ($\omega^*=k_{yv}0$):

$$\frac{AT_\perp}{m_i\omega^2}\frac{d^2\varphi}{dz^2}+\left(1+\frac{T_\perp}{T}-A-A\frac{\Omega}{\omega}\right)\varphi=\frac{1}{2b}\left(\frac{T_\perp}{T}-\frac{\Omega}{\omega-\omega^*}\right)\int_{-b}^{b}\varphi(z)\,dz. \tag{9}$$

For definiteness, consider the case in which the boundary condition has the form $\varphi_{z=\pm b}=0$. Since in this case the solution depends substantially on z, and moreover (according to the initial condition) $k_z\sim 1/b \gg \omega\sqrt{m_e/T}$, in the simplified theory (with equations borrowed from the theory for an infinite cylinder) the flute instability is absent, and only the drift instability exists. We shall see that, in the case of a short trap, the presence of an integral term in the right-hand side of (9), by no means small, leads to a flute (aperiodic) instability also for solutions that depend substantially on z. Let $\omega^*\ll\Omega$. Then, for $A\ll 1$ (waves short in the transverse direction), the dispersion equation

$$1-\frac{\mathrm{tg}\left[\frac{m_i\omega^2}{AT_\perp}\left(1+\frac{T_\perp}{T}-A-A\frac{\Omega}{\omega}\right)\right]^{1/2}b}{\left[\frac{-m_i\omega^2}{AT_\perp}\left(1+\frac{T_\perp}{T}-A-A\frac{\Omega}{\omega}\right)\right]^{1/2}b}=\frac{1+\frac{T_\perp}{T}-A-\frac{\Omega}{\omega}A}{\frac{T_\perp}{T}-\frac{\Omega}{\omega-\omega^*}} \tag{10}$$

indeed has complex roots if $4\omega^*\Omega m_i b^2 T_\perp^{-1} > 3A$. We note that, in contrast to the case of a long trap, in a short trap the flute instability affects short-wavelength ($k_\perp R_i \gg 1$) perturbations, while long waves are stabilized. This stabilization is due to averaging, over the length of the trap, of the field acting on the electrons, and in a certain sense is analogous to the Rosenbluth-Krall-Rostoker stabilization [1], associated with averaging, over the circular orbit, of the field acting on the ion.

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Note: Figure translations are in progress. See original paper for figures.