MATHEMATICS

L. A. PETROSYAN

1965

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. A. PETROSYAN

DIFFERENTIAL SURVIVAL GAMES WITH MANY PARTICIPANTS

(Presented by Academician Yu. V. Linnik, 8 X 1964)

The games considered are models of pursuit in a certain convex set S in the plane with several participants: a group of pursuers $P = \{P_1, \dots, P_n\}$ and pursued objects $E_1, \dots, E_n$, which move in the set S with constant linear speeds and have the ability at each moment of time to change the direction of their motion. A pursued object E_j is considered caught as soon as there is an $i \in \{1, 2, \dots, n\}$ such that the points P_i and E_j coincide. Each of the pursued objects is interested in reaching the boundary of the set S before being caught by one of the pursuers.

In what follows we shall use the definitions and notation of note ([1]).

1. The game $\Gamma(n,1)$**.** The game $\Gamma(n,1)$ is an antagonistic game of two players: a group of pursuers $P = \{P_1, \dots, P_n\}$ and a pursued object E. In the plane there is given a certain closed convex set S. Points $\xi^1, \dots, \xi^n, \eta \in S$, called the initial positions of the pursued $P_1, \dots, P_n$ and of the pursued object E, and $n+1$ numbers $v_1, \dots, v_n, u$, $v_i > u$ for all $i = 1, 2, \dots, n$, are given.

The structure of the game is specified by the system of equations

$$\dot{x}^i = \varphi^i[\psi(x^1, x^2, x^3, \dots, x^n, t)],$$

$$\dot{y} = \psi(x^1, x^2, \dots, x^n, t), \quad i = 1, 2, \dots, n,$$

with initial conditions $x^i(0) = \xi^i,\ y(0) = \eta,\ i = 1, 2, \dots, n.$

The set $\mathfrak{E}$ of strategies of E and the sets $\mathfrak{P}^i$ of strategies of P_i satisfy conditions 1–4 ([1]). From these conditions it follows directly that to each situation $(\varphi^1, \dots, \varphi^n, \psi)$ and initial conditions $\xi^1, \dots, \xi^n, \eta$ there correspond uniquely $2(n+1)$ continuous functions $x_1^i(t), x_2^i(t), y_1(t), y_2(t),\ i = 1, 2, \dots, n.$

The pair $\{x_1^i(t), x_2^i(t)\}$ will be called the trajectory of P_i. Let $t_P = \min_i (t_P)^i$. The payoff function is defined in the same way as in the game Γ.

Theorem 1. *The game* $\Gamma(n,1)$ *has an equilibrium situation in pure strategies. Moreover, the optimal strategy of the group* $P = \{P_1, \dots, P_n\}$ *consists in each* P_i *pursuing* E *according to the* Π*-strategy.*

Theorem 2. *Suppose that in the game* $\Gamma(n, 1)$ *there exists a circle* B*, belonging to* S*, of radius* a*, such that: a)* $\xi^1, \dots, \xi^n \in B$ *at the initial moment; b)* $\eta \in B$ *at the initial moment; c) we radially project the points corresponding to the positions of the pursuers at the initial moment onto the circumference of* B*; denote these projections by* $\xi_B^1, \xi_B^2, \dots, \xi_B^n$*, where the indices* $1, 2, \dots, n$ *increase in the clockwise direction; the distance between two neighboring projections must be less than* $a/3$*.*

If conditions a), b), c) are fulfilled, then E *is caught in* S *for any* $v_i/u \geq 1$*,* $i = 1, 2, \dots, n$*.*

2. The games Γ', Γ'_T, $\bar{\Gamma}'_T$, $\Gamma'(1, 2)$. The game Γ' differs from Γ only in the payoff function, which is defined as follows.

Let $y(t_P)$ be the point of capture in the situation (φ, ψ). Suppose that it belongs to S. Denote by $\rho[y(t_P), S]$ the distance from the point of capture to the boundary of S. Let $f(\rho)$ be a certain function having the following properties: a) $f(\rho)$ is a strictly decreasing function of ρ; b) let $0 < \varepsilon < 1$, then $f(0) = \varepsilon$; c) let ρ_0 be the distance from η to S; if capture occurs at the point η, then $f(\rho_0) = 0$.

The payoff function in the game Γ', $K'(\xi, \eta, \varphi, \psi)$, is defined through $f(\rho)$ as follows.

$$K'(\xi, \eta, \varphi, \psi) = \begin{cases} -1, & \text{if } t_P > t_{S_E}, \\ 1 - f(\rho), & \text{if } t_P \leqslant t_{S_E}, \\ 0, & \text{if } t_P = t_{S_E} = \infty. \end{cases}$$

Theorem 3. *In the game* Γ' *there exists an equilibrium situation in pure strategies. The player* P *has, moreover, a unique optimal strategy—the* P*-strategy.*

Consider a certain generalization Γ'_T of the game Γ', in which P can begin pursuit only after some time T has elapsed from the start of the game. The game Γ'_T obtained in this way will be called the game with delay T, or simply the game with delay.

Lemma 1. *In the game* Γ'_T *the* P*-strategy is optimal for* P*.*

Suppose further that in the time interval $[0, T]$ the player E may use only some subclass $\mathfrak{C}_T$ of his strategies. Denote this game by $\bar{\Gamma}'_T$.

Lemma 2. *In the game* $\bar{\Gamma}'_T$ *the* P*-strategy is optimal for* P*; for any* $\varepsilon > 0$*,* E *has an* ε*-optimal strategy.*

We now turn to the solution of the game $\Gamma'(1, 2)$. The game $\Gamma'(1, 2)$ is a three-person game: the pursuer P and the pursued E_1 and E_2. We shall assume that the players E_1 and E_2 are forbidden to form a coalition. The strategy spaces are the same as in the game Γ'. v is the linear velocity of the pursuer P; u_1

is the linear velocity of the pursued E_1; u_2 is the linear velocity of the pursued E_2.

The inequality $v > u_i,\ i = 1, 2$, holds. As in the game Γ', the strategies of the players are denoted by φ, ψ^1, ψ^2.

The payoff of $E_i,\ i = 1, 2$, is defined as the negative of the payoff of P in the game $\Gamma'(P, E_i)$. Let K_i be the payoff of $E_i,\ i = 1, 2$. Then the payoff of P is given by the expression $K_P = -K_1 - K_2$.

Thus the game $\Gamma'(1, 2)$ is a zero-sum game. In what follows, we shall denote by $\psi^i(t)$ the optimal strategy of E_i in the game $\Gamma'(P, E_i)$, beginning at time t, and by $\varphi^{i,3-i}$, $i = 1, 2$, the strategy of the pursuer P consisting in the fact that P successively applies the P-strategy first to E_i, and then to E_{3-i}.

Consider two situations $(\vec{\psi}^1, \vec{\psi}^2, \varphi^{1,2})$ and $(\vec{\psi}^1, \vec{\psi}^2, \varphi^{2,1})$. Let $K_P^{1,2}$ be the payoff in one situation, and $K_P^{2,1}$ in the other.

Theorem 4. *If $K_P^{1,2} \geqslant K_P^{2,1}$, then an ε-Nash equilibrium situation in the game $\Gamma'(1, 2)$ is constructed as follows:*

a) *the optimal strategy of P, $\varphi^{1,2}$, until the moment while the inequality*

$$K_P(\vec{\psi}^1, \vec{\psi}^2, \varphi^{1,2}) \geqslant K(\vec{\psi}^1, \psi^2, \varphi^{2,1}), \tag{1}$$

holds, and $\varphi^{2,1}$ as soon as E_2 begins to apply the strategy ψ^2 that violates (1);

b) *the optimal strategy of E_1, ψ^1;*

c) an ε-optimal strategy E_2 is defined as an ε-optimal strategy $\psi^{\varepsilon 2}$ in the game $\Gamma'_p(E_2, P)$ under the restrictions on the class of admissible strategies

$$K_P(\vec{\psi}^1, \psi^2, \varphi^{1,2}) \geqslant K_P(\vec{\psi}^1, \psi^2, \varphi^{2,1}), \tag{2}$$

$$K_{E_2}(\psi^1, \psi^2, \varphi^{1,2}) \geqslant K_{E_2}(\vec{\psi}^1, \psi^2, \varphi^{2,1}).$$

The game $\Gamma(m, n)$**.** Under the preceding conditions there are m pursuers $P_1, \dots, P_m$ and n evaders $E_1, \dots, E_n$. We shall denote this game by $\Gamma(m, n)$. For solving this game, so far, using methods of integer linear programming, it has been possible only to estimate from below the value of the game $\operatorname{Val}\Gamma(m, n)$. The estimate of $\operatorname{Val}\Gamma(m, n)$ is obtained as follows. Suppose that the set of pursuers $P_1, \dots, P_m$ is divided into n pursuing squads $M_1, \dots, M_n$ in such a way that, by the end of the game, each of the squads $M_1, \dots, M_n$ can pursue only one of the evaders E_i according to a P-strategy. For each fixed partition of the set of pursuers into squads $M_1, \dots, M_n$, the greatest payoff of the set of pursuers $f(M_1, \dots, M_n)$ is obtained as the solution of the following integer linear programming problem [2]:

$$\max \sum_{i,j}^{m,n} \xi_{ij} a_{ij} \tag{3}$$

under the conditions

$$\sum_{i=1}^{m} \xi_{ij} \leqslant 1, \quad \sum_{j=1}^{n} \xi_{ij} \leqslant 1$$

for all i, j (here a_{ij} is the payoff of the i-th squad M_i from pursuing E_j).

The evaders cannot enter into coalitions.
It is clear that

$$\operatorname{Val} \Gamma(m,n) \geqslant \max_{\{M_1,\dots,M_n\}} f(M_1,\dots,M_n), \tag{4}$$

where $\{M_1,\dots,M_n\}$ is the set of all possible partitions of $P_1,\dots,P_m$ into n groups.

Received
5 X 1964

CITED LITERATURE

[1] L. A. Petrosyan, DAN, **161**, No. 1 (1965).
[2] D. Gale, *The Theory of Linear Economic Models*, N. Y., 1960.

Note: Figure translations are in progress. See original paper for figures.