ON THE CONVERGENCE OF TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS TO NORMAL FORM

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Abstract

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ON THE CONVERGENCE OF TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS TO NORMAL FORM

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Let $\varphi_1(X), \ldots, \varphi_n(X)$ be power series in $x_1, \ldots, x_n$ without constant terms, convergent in some neighborhood of the point $X = 0$. Then $X = 0$ will be a singular point of the system of differential equations

$$dx_i/dt = \varphi_i(X), \qquad i = 1, \ldots, n. \tag{1}$$

Suppose further that $\xi_i(Y)$ and $\psi_i(Y)$, $(i = 1, \ldots, n)$, are power series in $y_1, \ldots, y_n$ without constant terms, convergent in some neighborhood of the point $Y = 0$, such that the transformation

$$x_i = \xi_i(Y), \qquad i = 1, \ldots, n, \tag{2}$$

fixed at zero, transforms system (1) into the system

$$dy_i/dt = \psi_i(Y), \qquad i = 1, \ldots, n. \tag{3}$$

In his dissertation ([1]), Poincaré proved that there exists a biholomorphic transformation at zero of the form (2), reducing system (1) to the form

$$dy_i/dt = \lambda_i y_i, \qquad i = 1, \ldots, n, \tag{4}$$

if the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\|\partial\varphi_i/\partial x_j|_0\|$ satisfy certain stringent conditions. The proof consists of two stages. First it is proved that there exist formal power series $\xi_i(Y)$ giving a formal transformation of system (1) to the form (4). The second stage consists in proving the convergence of the series $\xi_j(Y)$.

Let us write system (3) in the form

$$dy_i/dt = y_i g_i(Y) = y_i \sum_{Q \in N_i} g_{iQ} Y^Q, \qquad i = 1, \dots, n, \tag{5}$$

where $Q = (q_1, \dots, q_n)$, $Y^Q = y_1^{q_1} \cdots y_n^{q_n}$; $N_i = \{Q :$ integers $q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n \geq 0,\ q_i \geq -1,\ \sum_1^n q_k \geq 0\}$, $i = 1, \dots, n$. Denote by $\Lambda = (\lambda_1, \dots, \lambda_n)$ the vector of eigenvalues of the matrix $\|\partial\varphi_i/\partial x_j|_0\|$.

In my paper ([2]) the following theorem was proved:

There exists an invertible formal transformation (2) of system (1) into a system (3) such that: α) the matrix $\|\partial\psi_i/\partial y_j|_0\|$ is in Jordan normal form; β) when system (3) is written in the form (5), g_{iQ} differ from 0 only for those Q for which the scalar product $(Q, \Lambda) = 0$. In this case system (3) shall be called the normal form of system (1).

This theorem is the completion of a long series of works on normal form. In parallel, the following question was studied: for which normal forms (3) does the holomorphy of φ_i imply the holomorphy of the transformation (2)? It had long been shown that in some cases the series ξ_i diverge. For example, the system $dx_1/dt = x_1^2,\ dx_2/dt = x_2 - x_1$ is reduced to normal form

$$dy_1/dt = y_1^2, \quad dy_2/dt = y_2$$

by the transformation

$$x_1 = y_1, \quad x_2 = y_2 + \sum_{k=1}^{\infty} (k-1)!\, y_1^k,$$

which diverges for every $y_1 \neq 0$.

The convergence of transformations to normal form has been proved up to the present time in the following 4 cases (the conditions are imposed on the eigenvalues λ and on the normal form in the notation (5)):

a) Suppose the numbers $\lambda_1, \dots, \lambda_n$ are represented by points of the complex plane, and the convex hull of the points λ does not contain zero ([3]);

b) $n = 2,\ \lambda_1 = -\lambda_2 \neq 0$ and $g_1(k,k) = -g_2(k,k)$, $k = 1, 2, \dots$ (the Poincaré center case ([4]));

c) $n = 3,\ \lambda_1 = -\lambda_2 \neq 0$ are real, $\lambda_3 = i$, and $g_{1(k,k,0)} = -g_{2(k,k,0)},\ g_{3(k,k,0)} = 0,\ k = 1, 2, \dots$ (the hyperbolic case of a periodic solution of a Hamiltonian system ([5]));

d) for all integral $Q \neq 0$

$$|(Q,\Lambda)| > \left(\sum_1^n |q_i|\right)^{-\nu}, \quad \nu \geq n+1 \quad (^{6,9}).$$

Condition a) is necessary and sufficient for the equation $(Q,\Lambda) = 0$ to have only a finite number of solutions $Q \in N = N_1 \cup \dots \cup N_n$ and, for the remaining $Q \in N$,

$$|(Q,\Lambda)| > \varepsilon \sum_1^n |q_k|.$$

Therefore this condition is, in its way, limiting. The remaining cases of convergence can be generalized.

Condition 1. The normal form (5) of system (1) is such that, if $G_Q = (g_{1Q}, \dots, g_{nQ})$, then $G_Q = \mu_Q \Lambda + \nu_Q \bar{\Lambda}$, where μ_Q and ν_Q are complex numbers.

Generally speaking, the normal form is not determined uniquely by the given system.

Lemma 1. *If at least one normal form of a given system (1) satisfies condition 1, then every normal form of this system satisfies condition 1.*

Condition 2. There exists an $\varepsilon > 0$ such that, for every $Q \in N$, either $(Q,\Lambda) = 0$, or $|(Q,\Lambda)| > \varepsilon$.

Theorem 1. *If, for system (1), some normal form satisfies condition 1, Λ satisfies condition 2, and the $\varphi_i(X)$ are holomorphic at $X = 0$, then there exists a transformation (2), holomorphic at $Y = 0$, which brings system (1) to normal form.*

In cases b) and c), conditions 1 and 2 are fulfilled:

b) condition 1 is fulfilled, since $G_{(k,k)} = g_{1(k,k)} \lambda_1^{-1} \Lambda$, $k = 1, 2, \dots$; condition 2 is fulfilled, since $|(Q,\Lambda)| = |\lambda_1|\,|q_1 - q_2| \geq |\lambda_1|$, if $q_1 \neq q_2$.

c) condition 1 is fulfilled, since

$$G_{(k,k,0)} = \frac{1}{2} g_{1(k,k,0)} \lambda_1^{-1} (\Lambda + \bar{\Lambda}), \quad k = 1, 2, \dots;$$

condition 2 is fulfilled, since the equation $(Q,\Lambda) = \lambda_1(q_1 - q_2) + i q_3 = 0$ determines, in the real space (q_1, q_2, q_3), the line $q_1 = q_2$, $q_3 = 0$. The distance from this line to the points of the integer lattice not lying on it is bounded below.

Condition 3. Let $\omega(p) = \min |(Q,\Lambda)|$ over all such Q that $\sum |q_i| = p$, $(Q,\Lambda) \neq 0$, q_i are integers, and let $\omega_k = \omega(p_k)$ be the successive minima of $\omega(p)$ as p

increases from 1 to ∞, i.e. $p_1 = 1$, $p_k < p_{k+1}$, and $\omega_k > \omega_{k+1}$, and for every $p < p_{k+1}$ one necessarily has $\omega(p) \geq \omega(p_k)$. Then

$$\sum_{k=1}^{\infty} \frac{\ln \omega^{-1}(p_k)}{p_k} < \infty.$$

It is essential that those Q for which $(Q, \Lambda) = 0$ do not participate in the formation of the successive minima. A somewhat different notion of successive minima of linear forms was used by Voronoi for the purposes of generalizing the continued-fraction algorithm ([7]).

If $n = 2$, $\lambda = \lambda_1^{+1}\lambda_2$ is real and r_k is the denominator of the k-th convergent of the continued fraction of the number λ, then condition 3 is equivalent to the condition

$$\sum_{1}^{\infty} r_k^{-1} \ln r_{k+1} < \infty.$$

Lemma 2. For every n there exists a $c_n > 1$ such that $p_k > c_n^k \cdot 2^{-1/2}$, $k = 1, 2, \ldots$.

Theorem 2. If the system (1) has some normal form (4), Λ satisfies condition 3, and the $\varphi_i(X)$ are holomorphic at $X = 0$, then there exists a transformation (2), holomorphic at $Y = 0$, which brings system (1) to normal form.

In case), condition 3 is satisfied, the equation $(Q, \Lambda) = 0$ has no solutions in integers $Q \neq 0$, and the normal form is (4). Siegel' s condition in terms of $\omega(p)$ is $\omega(p_k) > p_k^{-\nu}$; then

$$\sum_{1}^{\infty} p_k^{-1} \ln \omega^{-1}(p_k) < \nu \sum_{1}^{\infty} p_k^{-1} \ln p_k.$$

The convergence of the latter series follows from Lemma 2.

It remains to note that the normal form (5), satisfying condition 1, is integrable. Let the integral vectors $R_1, \ldots, R_d$ form a basis in the lattice that is the intersection of the integral lattice with the solutions of the equation $(Q, \Lambda) = 0$. Then, first,

$$Y^{R_i} = \text{const} \equiv c_i \qquad (i = 1, \ldots, d)$$

are first integrals and, second,

$$g_i(Y) \equiv \sum_{(Q,\Lambda)=0} g_{iQ} Y^Q = \tilde{g}_i(Y^{R_1}, \ldots, Y^{R_d}), \quad i = 1, \ldots, n,$$

are functions only of the first integrals. Finally,

$$d \ln y_i = \tilde{g}_i(c_1, \dots, c_d)\, dt, \qquad i = 1, \dots, n$$

(see also § 2 of my paper [8]).

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Note: Figure translations are in progress. See original paper for figures.