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Abstract

Full Text

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ON THE ASYMPTOTIC PROPERTIES OF THE SPECTRAL FUNCTION OF HYPOELLIPTIC OPERATORS

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MATHEMATICS

1°. In this note we study the asymptotic behavior of the spectral function of hypoelliptic operators in n variables satisfying condition (2) (see § 2°). This class is considerably broader than that considered in paper ([1]). The sharper estimates of the fundamental solution used here make it possible to obtain results which, in particular, strengthen Theorem 2 of paper ([1]).

2°. We shall denote a point of n-dimensional real Euclidean space R^n by $x = (x_1, \dots, x_n)$, $|x| = \sqrt{x_1^2 + \dots + x_n^2}$; a point of n-dimensional complex space C^n by $s = \sigma + i\tau$, where $\sigma, \tau \in R^n$, $|s| = \sqrt{|\sigma|^2 + |\tau|^2}$. Let D be a domain in R^n, $C_0^k(D)$ the set of finite functions in D having k continuous derivatives, $C_0^\infty(D)$ the set of finite functions infinitely differentiable in D, and $C_0(D)$ the finite continuous functions in D.

Consider on functions $u \in C_0^\infty(D)$ the hypoelliptic differential operator

$$P\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right),$$

whose characteristic polynomial $P(s_1, \dots, s_n) = P(s)$ is complete (see ([2])) and has constant real coefficients. Denote by P_0 the minimal operator ([2]), and by $\hat{P}$ any semibounded self-adjoint extension of it. Let $\theta(x, y, \lambda)$ be the spectral function of the operator $\hat{P}$. If $D = R^n$, then the spectral function, as is known ([3]), is expressed by the formula

$$\theta_0(x - y, \lambda) = \frac{1}{(2\pi)^n} \int_{P(\sigma)<\lambda} \exp\left[-i \sum (x_k - y_k)\sigma_k\right] d\sigma_1 \cdots d\sigma_n. \qquad (1)$$

Finally, let λ_0 be a number less than the lower bounds of the operators $\hat{P}$ and P_0, defined on all of R^n.

Introduce the class K of operators satisfying, for every $c > 1$ and all sufficiently large λ, the inequality

$$\frac{\theta_0(0, c\lambda)}{\theta_0(0, \lambda)} \le c^\gamma \qquad (2)$$

with some $\gamma > 0$ (depending on the operator). This class is sufficiently broad*. In particular, it includes: a) all elliptic and quasi-elliptic operators; b) operators satisfying, for sufficiently large r, the condition

$$\frac{P_r'(r,\varphi)}{P(r,\varphi)} \geq \frac{A}{r},$$

* The question of whether there exist hypoelliptic operators not belonging to K remains open.

where $(r,\varphi) = (r,\varphi_1,\dots,\varphi_{n-1})$ are polar coordinates in R^n, and $A > 0$ does not depend on φ. It is easy to see that condition b) is satisfied, for example, by hypoelliptic polynomials in n variables with positive coefficients and even degrees $\sigma_1,\dots,\sigma_n$, in particular by the three-term operators considered in (1).

It can be shown that the class K also contains operators satisfying neither of the conditions a), b).

Let us formulate the main results obtained in the present paper.

Theorem 1. *If a hypoelliptic operator belongs to the class K, then as $\lambda \to +\infty$*

$$\theta(x,x,\lambda) \sim \theta_0(0,\lambda)$$

uniformly on compact subsets of the domain D.

Theorem 2. *For any hypoelliptic operator, as $\lambda \to +\infty$, the asymptotic estimate holds for every real $k \geqslant 1$*

$$I^k\theta(x,y,\lambda) = I^k\theta_0(x-y,\lambda) + o\left(\lambda^{(\beta-k\nu)/p}\right),$$

uniformly on each compact subset of the set $D \times D$.

Here $I^k\sigma(\lambda)$ denotes the Riesz mean of order k of the function σ on the interval (λ_0,λ), p is the degree of $P(s)$, and β and ν are positive constants depending on the operator ($\nu \leqslant 1$).

For the proof of these theorems we shall need a number of auxiliary assertions.

3°. Lemma. *For a given hypoelliptic polynomial $P(s)$ of degree p with real coefficients ($s_j = \sigma_j + i\tau_j$; $j = 1,\dots,n$), one can choose positive constants $a, b, h \leqslant p$ and $\nu \leqslant 1$ such that throughout the space C^n, for $t > 0$, the inequality*

$$\left|e^{-tP(s)}\right| \leqslant C \exp\left[-ta\sum_{j=1}^{n}|\sigma_j|^h + tb\sum_{j=1}^{n}|\tau_j|^{p/\nu}\right]. \tag{3}$$

Let $0 < \varepsilon < 1$. Using Taylor' s formula for the polynomial, we can write

$$|\exp[-tP(s)]|\exp[t(1-\varepsilon)P(\sigma)] = \exp\left[-\varepsilon tP(\sigma) - \sum_q \frac{D^{qP}(\sigma)}{q!}\tau^q\right], \quad (4)$$

where $D^q = \partial^{q_1+\dots+q_n}/\partial\sigma_1^{q_1}\cdots\partial\sigma_n^{q_n}$, $|q| = \sum q_j$ are even numbers; $\tau^q = \tau_1^{q_1}\cdots\tau_n^{q_n}$; $q! = q_1!\cdots q_n!$. Consider the region $S_{r_0}\{|\sigma| > r_0\}$, where r_0 is sufficiently large. Since for hypoelliptic operators with real coefficients, for large $|\sigma|$,

$$\min_{|\sigma|=r} P(\sigma) = a_1 r^h(1+o(1))$$

with some $a_1 > 0$, $h > 0$ (h rational) (see (4)), the left-hand side of equality (4) in this region can be estimated from below by the function $C_1|\exp[-tP(s)]|\exp[ta|\sigma|^h]$, $a > 0$. It can be shown that for each hypoelliptic operator there is such a $0 < \nu \leqslant 1$ that in the region $S_\nu\{|\tau| \leqslant c_1|\sigma|^\nu;\ |\sigma| > r_0\}$, for sufficiently large r_0, the right-hand side of (4) will be bounded above by some constant, and in the region $S_{r_0} - S_\nu$ by the function $C_2\exp[C_3 t|\tau|^{p/\nu}]$. Finally, if r_0 has already been chosen in the indicated way, then for $|\sigma| \leqslant r_0$, in view of the boundedness of $D^{qP}(\sigma)$, one easily obtains the estimate

$$|\exp[-tP(s)]| \leqslant C_4\exp[-at|\sigma|^h + C_5 t|\tau|^p].$$

Combining all these estimates into one and taking into account the elementary inequality

$$0 < m \leqslant |\xi|^l/\sum|\xi_k|^l \leqslant M, \qquad \xi \in R^n,$$

we obtain the assertion of the lemma.

4°. Consider the equation, parabolic in the sense of G. E. Shilov ([5]),

$$\frac{\partial u}{\partial t} = P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u,$$

where P is an arbitrary hypoelliptic operator with constant real coefficients. We derive estimates for the derivatives with respect to x and t of its fundamental solution

$$G_0(t, x_1, \dots, x_n) =$$

$$= \frac{1}{(2\pi)^n}\int_{R^n}\exp\left[-tP(\sigma_1,\dots,\sigma_n) - i\sum x_k\sigma_k\right]d\sigma_1\dots d\sigma_n, \quad (5)$$

using the device applied in ([5]) for functions of one variable. In view of inequality (3), by Cauchy' s theorem, the value of the integral will not change if one integrates along the horizontal straight lines $\tau_k = \alpha_k$ $(k = 1, \dots, n)$. In view of the separation of variables in estimate (3),

$$|D_x^q G_0(t,x)| \leq \prod_{k=1}^{n} \int_{-\infty}^{\infty} |s_k|^{q_k} \exp\left[-at|\sigma_k|^h + bt|\tau_k|^{p/\nu}\right] \exp[-x_k\tau_k]\, d\sigma_k.$$

For each of the factors we obtain

$$\int_{-\infty}^{\infty} \left(\sqrt{\sigma_k^2 + \tau_k^2}\right)^{q_k} \exp\left[-at|\sigma_k|^h + bt|\tau_k|^{p/\nu}\right] \exp[-x_k\tau_k]\, d\sigma_k \leq$$

$$\leq B_k \exp\left[-x_k\tau_k + tb|\tau_k|^{p/\nu}\right] \int_{-\infty}^{\infty} (|\sigma_k|^{q_k} + |\tau_k|^{q_k}) \exp\left[-at|\sigma_k|^h\right]\, d\sigma_k. \qquad (6)$$

Choose the quantity τ_k so that

$$|\tau_k| = \left(|x_k|\, \frac{pb}{\nu}\, t\right)^{\nu/(p-\nu)},$$

and its sign so that $x_k\tau_k = |x_k| \cdot |\tau_k|$. Substituting τ_k in the right-hand side of (6) and multiplying the estimates, we obtain

$$|D_x^q G_0(t,x)| \leq \prod_{k=1}^{n} B_k \left(\frac{C_k'}{t^{(1+q_k)/h}} + \frac{|x_k|^{q_k\nu/(p-\nu)}}{t^{q_k\nu/(p-\nu)+1/h}}\right) \exp\left[-C_k' \frac{|x_k|^{p/(p-\nu)}}{t^{\nu/(p-\nu)}}\right]. \quad (7)$$

Analogous estimates are also obtained for derivatives with respect to t. Since for hypoelliptic operators $0 < \nu \leq 1$, $p \geq 2$, it follows that $\nu(p-\nu) < 0$.

5°. Consider in $C_0(D)$ the Hermitian bilinear form

$$V(t,f,g) = ((e^{-t\hat{P}} - e^{-tP_0})f, g).$$

Its kernel, in view of the formula

$$(E_\lambda f, g) = \int_{D\times D} \theta(x,y,\lambda) f(x) g(y)\, dx\, dy$$

is the difference

$$v(t,x,y) = G(t,x,y) - G_0(t,x-y), \tag{8}$$

where $G(t,x,y)$ is the Laplace-Stieltjes transform of the spectral function $\theta(x,y,\lambda)$ of the operator $\hat{P}$ (the existence of $G(t,x,y)$ follows from Theorem 3.9 of Hermander ([2])). Let Q be a fixed compact subset of the domain D. In view of the boundedness of the kernel $e^{-t\lambda}$, for any $t>0$ and any $f,g \in C_0(Q)$,

$$|V(t,f,g)| \leq C_Q \sup_{x\in Q}|f(x)| \sup_{x\in Q}|g(x)|, \tag{9}$$

where C_Q does not depend on t. Let W be any compact subset of the domain D. Choose an open set $U \subset D$ with compact closure, containing W. Further, let $x \in U$ and let F be a compact subset of D containing $\overline{U}$. Take $h(x) \in C_0^\infty(D)$ with support F, $h(x) = 1$ on U, and construct the function

$$b(t,x,y) = \left[P\left(\frac{1}{i}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)\right]\left[G_0(t,x-y)(1-h(y))\right]. \tag{10}$$

Obviously, $b(t,x,y)$ is a finite function in y with support contained in $Q = F \setminus U$ for every $x \in U$, and $b(t,x,y) \to 0$ as $t \downarrow 0$ for each $(x,y) \in U \times D$. We apply Gårding' s identity ([3]), which in our notation (8) and (10) has the form

$$v(t,x,y) = \int_0^t dt' \int_0^{t'} dt'' \int_{D\times D} v(t'',\xi,\eta) b(t-t',x,\xi) b(t'-t'',x,\eta)\, d\xi\, d\eta \tag{11}$$

and is valid for any $(x,y) \in U \times U$.

By virtue of inequality (9) and identity (11), for $0 < t < t_0$,

$$|v(t,x,y)| \leq C_Q t^2 \sup_{0\leq\tau\leq t}\sup_{\xi\in Q}|b(\tau,x,\xi)| \sup_{0\leq\tau\leq t}\sup_{\eta\in Q}|b(\tau,y,\eta)|.$$

Using estimates (7), the last inequality and the fact that, for all points $x \in W$ and $\xi \in Q$, $0 < c' \leq |x-\xi| \leq C'$, we obtain, as $t \downarrow 0$, the asymptotic estimate

$$|G(t,x,y) - G_0(t,x-y)| = o\left(\exp\left[-\frac{C}{t^{\nu/(p-\nu)}}\right]\right), \tag{12}$$

uniformly in $(x,y) \in W \times W$, with some constant $C > 0$.

6°. We pass to the direct proof of the theorems formulated in Section 2°. Theorem 1 follows immediately from estimate (12) and B. I. Korenblyum's Tauberian theorem for ratios of functions ([6]).

For the proof of Theorem 2, Ganelius' Tauberian theorem is applied to the function $\sigma(\lambda) = \theta(x, y, \lambda) - \theta_0(x - y, \lambda)$, with remainder estimate (7). Put $\Lambda = (\lambda, \lambda + \lambda^{(p-\nu)/p})$. From the relations

$$\sup_{\mu \in \Lambda} \int_\lambda^\mu d\sigma(\omega) \leq \operatorname{Var}_\Lambda \sigma(\lambda) \leq \operatorname{Var}_\Lambda \theta(x, y, \lambda) + \operatorname{Var}_\Lambda \theta_0(x - y, \lambda),$$

$$\operatorname{Var}_\Lambda \theta(x, y, \lambda) \leq [\operatorname{Var}_\Lambda \theta(x, x, \lambda) \operatorname{Var}_\Lambda \theta(y, y, \lambda)]^{1/2}$$

it follows that the Tauberian condition of the theorem can be obtained by estimating the variations of the functions $\theta(x, x, \lambda)$ and $\theta_0(0, \lambda)$. Because these functions are nondecreasing, we obtain the rough estimate

$$\sup_{\mu \in \Lambda} \int_\lambda^\mu d\sigma(\omega) = o(\lambda^\alpha),$$

where α is some positive exponent depending on the operator. Now Ganelius' theorem gives

$$I^k \sigma(\lambda) = o\left(\lambda^{\alpha + (\nu - p)/p} (\lambda - \lambda_0) \frac{\lambda^{k(p-\nu)/p}}{(\lambda - \lambda_0)^k}\right) = o\left(\lambda^{\beta - k\nu/p}\right),$$

which was to be proved. Since $\nu/p > 0$, for sufficiently large k the Riesz mean of order k of the function $\sigma(\lambda)$ tends to zero uniformly in $(x, y) \in W \times W$.

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Note: Figure translations are in progress. See original paper for figures.