D. L. BERMAN

MATHEMATICS

1965

SovietRxiv

Abstract

Full Text

D. L. BERMAN

THE S. N. BERNSTEIN INTERPOLATION PROCESS IN THE COMPLEX DOMAIN

(Presented by Academician S. N. Bernstein on 24 II 1965)

MATHEMATICS

1°. Denote by C the set of all functions $f(x)$ continuous on the segment $[-1, 1]$. By $L_n(f, x)$ denote the Lagrange interpolation polynomial of degree n, constructed for the $(n+1)$ rows of the matrix of nodes

$$\begin{gathered} x_1^{(1)} \\ x_1^{(2)} x_2^{(2)} \\ \dots \\ -1 \le x_1^{(n)} < x_2^{(n)} < \dots < x_n^{(n)} \le 1, \quad n = 1, 2. \end{gathered} \tag{1}$$

The classical Bernstein-Faber theorem (^{1,2}) asserts that there is no matrix of nodes of the form (1) for which, for every $f \in C$, the relation

$$L_n(f, x) \to f(x), \qquad n \to \infty$$

holds uniformly.

For the case of Chebyshev nodes

$$x_k^{(n)} = \cos[(2n - 2k + 1)\pi/2n],$$

$k = 1, 2, \dots, n$, which in the theory of interpolation of functions of a real variable are in a certain sense the best possible, G. Grünwald (^{3}) and I. Marcinkiewicz (^{4}) constructed such an $f \in C$ that at every point $x \in [-1, 1]$ the equality

$$\overline{\lim}_{n\to\infty} L_n(f, x) = \infty$$

holds. In connection with the Bernstein-Faber theorem there naturally arose the question of replacing the Lagrange interpolation process by another interpolation process $\{A_n(f, x)\}_{n=1}^{\infty}$ which, already for every $f \in C$, uniformly satisfies the relation

$$A_n(f, x) \to f(x), \qquad n \to \infty.$$

This problem was solved by S. N. Bernstein and L. Fejér (^{5}). The solution of S. N. Bernstein (^{6}) is remarkable in that it is obtained by a simple modification of the Lagrange interpolation formula, and the ratio of the degree of the interpolation polynomial $A_n(f,x)$ to the number of its nodes can be made arbitrarily close to one. The interpolation polynomials $A_n(f,x)$ of S. N. Bernstein were also studied in works ($^{7-10}$). Until now the S. N. Bernstein interpolation polynomials have been studied only in the real domain. In this note they are studied in the complex domain.

Denote by A the set of all functions $f(z)$ analytic inside the disk $|z|<1$ and continuous in the closed disk $|z|\le 1$. Introduce in A the norm according to the equality

$$\|f\| = \max_{|z|\le 1} |f(z)|.$$

Obviously, A is a Banach space.

For simplicity we shall consider the special case of the S. N. Bernstein interpolation polynomials $A_n(f,z)$ (^{6}), when

$$A_n(f,z) = \sum_{k=1}^{m} f(z_{2k-1}^{(n)})\,[\,l_{2k-1}^{(n)}(z) + l_{2k}^{(n)}(z)\,], \quad n = 2m, \tag{2}$$

where $\{l_j^{(n)}(z)\}_{j=1}^{n}$ are the fundamental Lagrange interpolation polynomials.

2°. First of all, by means of very simple estimates we shall prove the following theorem.

Theorem 1. The polynomials $A_n(f,z)$ constructed at the nodes

$$z_k^{(n)} = e^{i2k\pi/n}, \quad k = 1,2,\dots,n; \quad n = 1,2,\dots, \tag{3}$$

for $f\in A$, converge uniformly inside the circle $|z|<1$ to $f(z)$.

Proof. It is obvious that

$$A_n(f,z) = 2\sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) l_{2k-1}^{(n)}(z) + \sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right)\left[l_{2k}^{(n)}(z) - l_{2k-1}^{(n)}(z)\right]. \tag{4}$$

It is easy to see that, for the nodes (3), the fundamental Lagrange interpolation polynomials $l_k^{(n)}(x)$ have the form

$$l_k^{(n)}(z) = \frac{(1-z^n)z_k}{n(z_k - z)}, \quad k = 1,2,\dots,n. \tag{5}$$

Consequently,

$$\sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) l_{2k-1}^{(n)}(z) = \sum_{k=1}^{m} f\left(e^{i2(2k-1)\pi/n}\right) \frac{1-z^n}{e^{i2(2k-1)\pi/n}-z} \frac{e^{i2(2k-1)\pi/n}}{n}.$$

Hence, as $n \to \infty$, by Cauchy' s formula we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) l_{2k-1}^{(n)}(z) = \frac{1}{4\pi i}\int_{\Gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \frac{f(z)}{2}, \quad |z| < 1. \tag{6}$$

From equality (5) it is immediately seen that

$$|l_{2k}(z) - l_{2k-1}(z)| \leq \frac{1+|z|^n}{n(1-|z|)^2} |z_{2k} - z_{2k-1}|.$$

Therefore, since $\sum_{k=1}^{m} |z_{2k} - z_{2k-1}| \leq 2\pi$, we have

$$\left|\sum_{k=1}^{m} f(z_{2k-1})[l_{2k}(z) - l_{2k-1}(z)]\right| \leq \frac{4\pi \max_{|z|\leq 1} |f(z)|}{n(1-|z|)^2}. \tag{7}$$

As a consequence of (4), (6), (7) we have

$$\lim_{n\to\infty} A_n(f,z) = f(z), \quad |z| < 1.$$

By Vitali' s theorem, the sequence $\{A_n(f,z)\}$ converges inside the circle $|z| < 1$ uniformly. From equality (5) it is seen that the assumption that n is even is not essential.

3°. Theorem 1 admits the following generalization. Let D be a bounded continuum of the z-plane such that its complement D_1 is a simply connected domain. We shall assume that the contour Γ of the domain D is a closed analytic Jordan curve. In fact, these conditions can be substantially weakened. By $w = \Phi(z)$ we denote the function mapping D_1 conformally onto the domain $|w| > 1$ of the w-plane under the condition $\Phi(\infty) = \infty$. Let $z = \psi(w)$ be the inverse function. As is known,

$$\psi(w) = cw + c_0 + c_1/w + \cdots \quad (c \neq 0).$$

Let the points

$$z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)}, \quad n = 1, 2, \ldots, \tag{8}$$

be situated on the curve Γ. According to L. Fejér [11], the points (8) are called regularly situated on the curve Γ if the equalities

$$z_k^{(n)} = \psi\left(w_k^{(n)}\right), \quad w_k^{(n)} = e^{i2\pi k/n}, \quad k = 1, 2, \ldots, n; \quad n = 1, 2, \ldots$$

are satisfied.

Theorem 2. Let the points (8) be regularly situated on the curve Γ and let $f(z)$ be R-integrable on Γ. Then the polynomials (2) satisfy, for any point $z \in D$, the relation

$$A_n(f, z) \to \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{\zeta - z}, \qquad n \to \infty. \tag{9}$$

The convergence is uniform in every closed domain situated inside D.

In the course of the proof of this theorem the following lemma was used:

Lemma. Under the hypotheses of Theorem 2, at every point $z \in D$,

$$\sum_{k=1}^{m} \left| l_{2k}^{(n)}(z) - l_{2k-1}^{(n)}(z) \right| \to 0, \qquad n \to \infty, \qquad n = 2m. \tag{10}$$

Relation (10) holds uniformly in every domain situated inside D.

For the proof of the lemma, let us note that

$$l_{2k}^{(n)}(z) - l_{2k-1}^{(n)}(z) = \frac{\omega_n(z)}{c^n} \, \frac{1}{in(z - z_{2k})(z - z_{2k-1})} \times$$

$$\times \left[z \left(\frac{\delta_{n,2k}}{\gamma_{n,2k}} - \frac{\delta_{n,2k-1}}{\gamma_{n,2k-1}} \right) + \left(z_{2k} \frac{\delta_{n,2k-1}}{\gamma_{n,2k-1}} - z_{2k-1} \frac{\delta_{n,2k}}{\gamma_{n,2k}} \right) \right];$$

$$\gamma_{n,k} = \frac{\omega_n'(z_k)}{inc^n}\, \delta_{n,k}; \qquad \delta_{n,k} = \frac{d\psi\left(w_k^{(n)}\right)}{d\varphi};$$

$$w = e^{i\varphi}; \qquad \omega_n(z) = \prod_{k=1}^{n} \left(z - z_k^{(n)}\right),$$

and use the following known facts:

1) In the case of an analytic curve Γ, the modulus of the derivative of the mapping function $\Phi(z)$ is bounded on Γ above and below by positive constants (^{12,13}).

2) For regularly distributed nodes (^{14})

$$\omega_n(z)/(-c^n) \to 1, \qquad n \to \infty. \tag{11}$$

Relation (11) holds uniformly in the domain $B \subset D$.

3) The relation $(^{15})$

$$\gamma_{n,k} \to 1, \qquad n \to \infty \tag{12}$$

holds uniformly with respect to k.

With the help of these facts and simple calculations, the lemma is obtained.

Proof of Theorem 2. By virtue of the lemma and equality (4),

$$\lim_{n\to\infty} A_n(f,z) = 2 \lim_{n\to\infty} \sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) l_{2k-1}^{(n)}(z), \qquad z \in D. \tag{13}$$

But

$$\sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) l_{2k-1}^{(n)}(z) = \frac{1}{4\pi i} \sum_{k=1}^{m} \frac{f\left[\psi\left(w_{2k-1}^{(n)}\right)\right]}{z-\psi\left(w_{2k-1}^{(n)}\right)} \, \frac{\omega_n(z)}{c^n} \, \frac{\delta_{k,\,2k-1}}{\gamma_{n,\,2k-1}} \, \frac{4\pi}{n}.$$

Therefore, by virtue of relations (11) and (12), we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) l_{2k-1}^{(n)}(z) = \frac{1}{4\pi i} \int_{\Gamma} \frac{f(\zeta)\, d\zeta}{\zeta - z}. \tag{14}$$

From (13) and (14), (9) follows.

4°. On the boundary of the domain the polynomials (2) may diverge. This is seen from the theorem:

Theorem 3. *Suppose that the points*

$$z_k^{(n)} = e^{i\theta_k}, \quad \theta_k = (2k-1)\pi/n, \quad k = 1,2,\dots,n; \quad n = 1,2,\dots \tag{15}$$

are taken as nodes.

Then there exist such $f \in A$ for which the polynomials (2) diverge at the point $z = 1$.

We outline the proof. For the nodes (15) we have

$$A_n(f,1) = \frac{2}{n} \sum_{k=1}^{m} f\left(z_{2k-1}^{(n)}\right) \left(\frac{z_{2k}^{(n)}}{z_{2k}^{(n)} - 1} + \frac{z_{2k-1}^{(n)}}{z_{2k-1}^{(n)} - 1} \right), \qquad n = 2m.$$

Therefore it is sufficient to prove that $\lim\limits_{n\to\infty} \lambda_n = \infty$, where

$$\lambda_n = \frac{2}{n} \sum_{k=1}^{m} \left| \frac{z_{2k}^{(n)}}{z_{2k}^{(n)} - 1} + \frac{z_{2k-1}^{(n)}}{z_{2k-1}^{(n)} - 1} \right|.$$

From this we obtain that

$$\lambda_n > \sum_{k=1}^{[(n-2)/4]} \frac{1}{n \sin(4k-1)\pi/2n} - 1 > \frac{1}{2\pi} \ln n + O(1).$$

Leningrad Institute of Soviet Trade
named after Fr. Engels

Received
16 I 1965

REFERENCES

[1] S. N. Bernstein, *Collected Works*, 1, Publishing House of the Academy of Sciences of the USSR, 1952, p. 253.
[2] G. Faber, Jahresber. DMV, **23**, 192 (1914).
[3] G. Grünwald, Ann. Math., **37**, 908 (1936).
[4] J. Marcinkiewicz, Acta Szeged, **8**, 134 (1937).
[5] L. Fejér, Gött. Nachr., 66 (1916).
[6] S. N. Bernstein, *Collected Works*, 2, Publishing House of the Academy of Sciences of the USSR, 1954, p. 130.
[7] D. L. Berman, DAN, **60**, No. 3 (1948).
[8] D. L. Berman, DAN, **70**, No. 2 (1950).
[9] D. L. Berman, DAN, **81**, No. 1 (1951).
[10] D. L. Berman, DAN, **101**, No. 3 (1955).
[11] L. Fejér, Gött. Nachr., 319 (1918).
[12] S. Warschawski, Math. Zs., **35**, 321 (1932).
[13] B. K. Dzyadyk, Izv. AN SSSR, Ser. Mat., **23**, 697 (1959).
[14] J. Curtiss, Trans. Am. Math. Soc., **38**, 458 (1935).
[15] D. Gaier, Math. Zs., **61**, 119 (1954).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.