ON THE \(S\)-MATRIX IN THE HEISENBERG REPRESENTATION

PHYSICS

1965

SovietRxiv

Abstract

Full Text

UDC 530.145.1

PHYSICS

B. V. MEDVEDEV, A. D. SUKHANOV

ON THE S-MATRIX IN THE HEISENBERG REPRESENTATION

(Presented by Academician N. N. Bogolyubov, March 18, 1965)

1. With the development of axiomatic methods in quantum field theory, representations of the S-matrix in the form of functional expansions in the operators of asymptotic fields $\varphi_{in}(x)$ (or $\varphi_{out}(x)$) have become widespread (^{1,2}). At the same time, in (^{3}), outside the framework of perturbation theory, the possibility was substantiated of representing the S-matrix in the form

$$S_{in} = T_W \exp\left\{ i \int_{-\infty}^{\infty} \mathcal{L}_{in}(z)\, dz \right\}, \tag{1}$$

where T_W is the Wick T-product (^{3,4}), defined by the expansion according to Wick' s theorem, and the operator $\mathcal{L}_{in}(z)$ has the meaning of an effective interaction Lagrangian (with counterterms) (^{5}), expressed through normal products of the operators $\varphi_{in}(x)$ and their derivatives.

Of special interest is the derivation of the S-matrix directly in the Heisenberg representation, since the corresponding results (see, for example, (^{6})) have not received due circulation. Below we propose a consistent method for obtaining the S-matrix in the Heisenberg representation, based on the assumption of the existence of the so-called "half" S-matrix*.

2. As the starting point it is natural to choose the *in*-representation of the S-matrix in the form (1), and, by the very meaning of the introduction of the S-matrix, it carries out (^{6}) a unitary transformation to a new –*out*-representation, for example, for the field operators

$$\varphi_{out}(x) = S_{in}^{+} \varphi_{in}(x) S_{in}; \qquad \varphi_{in}(x) = S_{in} \varphi_{out}(x) S_{in}^{+}, \tag{2}$$

and similarly for the others. Since we construct the theory on the basis of a number of general axioms (^{2,3}), we shall obtain the "half" S-matrix $S_{in}(\sigma, -\infty)$ not in the usual way, by solving the Tomonaga–Schwinger equation, but directly from the matrix S_{in}, guided by the requirements: a) relativistic covariance; b) independence of the specific choice of the spacelike surface σ; c) finiteness; d) unitarity:

$$S_{in}^{+}(\sigma,-\infty)S_{in}(\sigma,-\infty)=S_{in}(\sigma,-\infty)S_{in}^{+}(\sigma,-\infty)=1; \tag{3}$$

e) fulfillment of the group property:

$$S_{in}=S_{in}(\infty,\sigma)S_{in}(\sigma,-\infty), \tag{4}$$

f) boundary conditions:

$$\lim_{\sigma\to\infty}S_{in}(\sigma,-\infty)=S_{in};\qquad \lim_{\sigma\to-\infty}S_{in}(\sigma,-\infty)=1. \tag{5}$$

The simplest possibility for obtaining a matrix $S_{in}(\sigma,-\infty)$, satisfying the enumerated requirements, from S_{in} of the form (1) consists in the following: perform the functional substitution $\mathcal{L}_{in}(z)$ by $\theta(\sigma(x)-z^0)\mathcal{L}_{in}(z)$. However, as has already been shown in perturbation theory (^{7,4}), the half S-matrix obtained in this way will not satisfy requirements c), d), and e), if only $\mathcal{L}_{in}(z)$ contains time derivatives of fields higher than first, i.e., in all renormalized theories (counterterms!). At the same time, in (^{4}) it was shown that the indicated difficulties in the definition of $S_{in}(\sigma,-\infty)$ can be avoided if one first passes, in the expression for S_{in} of the type (1), from the T_W-product to the T_D-product (Dyson T-product, defined by explicit chronological ordering (^{3,4})) and simultaneously from $\mathcal{L}_{in}(z)$ to $H_{in}(z;\sigma)$, the interaction Hamiltonian $H_{in}(z;\sigma)$ being obtainable, according to definite rules, from $\mathcal{L}_{in}(z)$ (^{4,8}).

Since in the present case we do not have at our disposal a concrete expression for $\mathcal{L}_{in}(z)$, relying on the analogy with perturbation theory (^{4}), we shall suppose that the matrix S_{in} of the form (1) can be represented in the form*

$$S_{in}=T_D\exp\left\{-i\int_{-\infty}^{\infty}H_{in}(z;\sigma)\,dz\right\}=S_{in}[H_{in}(z;\sigma)], \tag{6}$$

where, generally speaking, $H_{in}(z;\sigma)\neq-\mathcal{L}_{in}(z)$.

Then the "half" S-matrix, if it exists at all, may be defined by the relation**

$$S_{in}(\sigma,-\infty)=S_{in}[\theta(\sigma(x)-z^0)H_{in}(z;\sigma')]=T_D\exp\left\{-i\int_{-\infty}^{\sigma}H_{in}(z;\sigma')\,dz\right\}, \tag{7}$$

---

* The results were reported at the Fifth All-Union Conference on the Theory of Elementary Particles (Uzhgorod, October 1963).

which, as is known from perturbation theory (^{4}), in renormalizable theories can satisfy all requirements a)—e), in connection with which below, for simplicity, we shall put $\sigma(x) = x^0 = \text{const}$.

3. We shall now pass to the out-representation. By the general rule for transforming operators, one must have

$$S_{out} = S_{in}^{+} S_{in} S_{in}, \tag{8}$$

and, since under the transformation (2) products are transformed in the same way, and consequently also polynomials and series of operators, it follows that

$$S_{out} = S_{in}[H_{out}(z)], \tag{9}$$

where $H_{out}(z) = S_{in}^{+} H_{in}(z) S_{in}$, while the notation in (9) means that the functional dependence (in the present case the T_D-product) of S_{out} on $H_{out}(z)$ is exactly the same as that of S_{in} on $H_{in}(z)$ in (6). However, the meaning of the operator S_{out} is not yet clear. If, moreover, we take into account in (8) the unitarity of S_{in}, it turns out that

$$S_{out} = S_{in} = S, \tag{10}$$

i.e., the value of S_{out} coincides with the value of S_{in}.

At the same time,

$$S_{out}(x^0, -\infty) = S^{+} S_{in}(x^0, -\infty) S = T_D \exp\left\{ -i \int_{-\infty}^{x^0} H_{out}(z)\, dz \right\}. \tag{11}$$

Thus, the matrix $S_{out}(x^0, -\infty)$ differs from the matrix $S_{in}(x^0, -\infty)$, although its functional dependence on $H_{out}(z)$ has remained

---

* The interaction Hamiltonian used by us should, more properly, be denoted $H_3^v(z;\sigma)$, since it is, generally speaking, distinct from the in-image of the interaction Hamiltonian (^{4}). Our notation is intended to emphasize that this operator depends precisely on the fields $\varphi_{in}(z)$, and not on $\varphi_{out}(z)$ or on anything else. Further, since in our case the free operators are $\varphi_{in}(x)$, the matrix $S_{in}(\sigma, -\infty)$ has no relation to the evolution operator $U(\sigma, -\infty)$, but coincides with the matrix $V_{+}(\sigma)$ in the notation of (^{9}).

** Such a definition, naturally, contains a factor of the form $\exp[iF_{in}(\sigma)]$, where $F_{in}(\sigma)$ is a Hermitian local operator which, from the point of view of the complete S-matrix, is immaterial.

same (T_D-product). Similarly, for the other "half" of the S-matrix we have

$$S_{out}(\infty, x^0) = S^+ S_{in}(\infty, x^0) S = S^+_{in}(x^0, -\infty) S. \tag{12}$$

However, the group property (4) is still preserved in the out-representation, i.e.

$$S_{out}(\infty, x^0) S_{out}(x^0, -\infty) = S^+ S_{in}(\infty, x^0) S S^+ S_{in}(x^0, -\infty) S = S. \tag{13}$$

Let us note that from (12) there follows a noteworthy formula, on which the subsequent transformations are based:

$$S_{in}(x^0, -\infty) S_{out}(\infty, x^0) = S. \tag{14}$$

Thus, for the complete scattering matrix there are three expressions in terms of "halves," namely (4), (13), and (14).

Let us now introduce, in addition to the in- and out-representations, a collection of j-representations ($j = 1, \dots, n$), according to the formula

$$A_j(z) = S^+_{in}(x^0_j, -\infty)\, A_{in}(z)\, S_{in}(x^0_j, -\infty), \tag{15}$$

where $A_{in}(z)$ is an arbitrary in-operator. Then

$$S_j(\infty, x^0_j) = S^+_{in}(x^0_j, -\infty) S; \qquad S_j(x^0_j, -\infty) = S_{in}(x^0_j, -\infty), \tag{16}$$

where the latter equality denotes not only the same functional dependence, but also coincidence of the operators.

Let us note that the matrix $S_j = S_j(\infty, x^0_j) S_j(x^0_j, -\infty)$ is not equal to the matrix S, but is related to it by the formula

$$S_j = S^+_{in}(x^0_j, -\infty) S S_{in}(x^0_j, -\infty), \tag{17}$$

i.e. it has only the same functional dependence. At the same time, from the first equality in (16) we obtain

$$S = S_{in}(x^0_j, -\infty) S_j(\infty, x^0_j). \tag{18}$$

Considering now simultaneously two such representations i and j (taking $i > j$), we obtain, with allowance for the fact that all transformations (15) form a group, that

$$A_i(z) = S^+_j(x^0_i, x^0_j) A_j(z) S_j(x^0_i, x^0_j), \tag{19}$$

where

$$S_j(x_i^0, x_j^0) = S_{in}^{+}(x_j^0, -\infty) S_{in}(x_i^0, x_j^0) S_{in}(x_j^0, -\infty). \tag{20}$$

Applying next the transformation (19) to the matrix $S_j(\infty, x_j^0)$, we obtain the relation

$$S_j(\infty, x_j^0) = S_j(x_i^0, x_j^0) S_i(\infty, x_i^0), \tag{21}$$

which, in combination with (18), gives for the complete S-matrix the expression

$$S = S_{in}(x_j^0, -\infty) S_j(x_i^0, x_j^0) S_i(\infty, x_i^0). \tag{22}$$

Carrying out these arguments now for all j successively, beginning with the first up to n, we arrive at

$$S = S_{in}(x_1^0, -\infty) \prod_{j=1}^{n-1} S_j(x_{j+1}^0, x_j^0) S_n(\infty, x_n^0). \tag{23}$$

Writing (18) for $j = n$ and comparing the expression obtained with (23), we also see that

$$S_{in}(x_n^0, -\infty) = S_{in}(x_1^0, -\infty) \prod_{j=1}^{n-1} S_j(x_{j+1}^0, x_j^0). \tag{24}$$

To pass to the expressions of interest to us, it is necessary in (23) and (24) to carry out the limiting transition $n \to \infty$ in such a way that, in doing so, $x_1^0 \to -\infty$, $\Delta_j = \max(x_{j+1}^0 - x_j^0) \to 0$, while $x_n^0 \to \infty$ in (23) and $x_n^0 \to x^0 = \text{const}$ in (24). Let us first note that the expression $S_{in}(x_{j+1}^0, x_j^0)$ entering into $S_j(x_{j+1}^0, x_j^0)$, according to (20), with account of (7), in the limit $\Delta_j \to 0$, if only terms of first order of smallness in Δ_j are retained, gives

$$\lim_{\Delta_j \to 0} S_{in}(x_{j+1}^0, x_j^0) = 1 - i \int_{x_j^0}^{{x_{j+1}}^{0}} H_{in}(z)\, dz. \tag{25}$$

Substituting now (25) into (20), and (20) into (23) and (24), we obtain for the S-matrix, in the indicated limit, an expression in the form of a Dyson antichronological exponential

$$S = \widetilde{T}_D \exp\left\{ -i \int_{-\infty}^{\infty} \mathbf{H}_z(z)\, dz \right\} \tag{26}$$

of Hamiltonians taken, for each point z, in its own special representation, determined by the "half" S-matrix with the separating surface passing through this point z:

$$H_{(z)}(z) = S^{+}_{in}(z^0, -\infty)\, H_{in}(z)\, S_{in}(z^0, -\infty). \tag{27}$$

But it is easy to see that such an infinite collection of representations forms precisely the Heisenberg representation

$$H_z = S^{+}_{in}(z^0, -\infty)\, H_{in}(z)\, S_{in}(z^0, -\infty), \tag{28}$$

i.e. formula (26) means that

$$S = S_{in} = S_{out} = S_\Gamma = T_D \exp\left\{-i \int_{-\infty}^{\infty} \mathbf{H}(z)\, dz\right\}; \tag{29}$$

and analogously from (24) we obtain

$$S_{in}(x^0, -\infty) = S_\Gamma(x^0, -\infty) = \widetilde{T}_D \exp\left\{-i \int_{-\infty}^{x^0} \mathbf{H}(z)\, dz\right\}. \tag{30}$$

Thus, in the transition from the *in*-representation to the Heisenberg one, the functional dependence in the S-matrix changes; namely, chronological ordering passes into antichronological ordering.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Moscow Institute
of Radio Electronics and Mining Electromechanics

Received
5 III 1965

CITED LITERATURE

1. H. Lehmann, K. Symanzik, W. Zimmermann, *Nuovo Cim.*, **1**, 205 (1955); **6**, 319 (1957).

2. N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov, *Problems in the Theory of Dispersion Relations*, Moscow, 1958; B. V. Medvedev, *ZhETF*, **40**, 826 (1961).

3. B. V. Medvedev, M. K. Polivanov, *Lectures at the International Winter School of Theoretical Physics*, Dubna, March 1964.

4. A. D. Sukhanov, *ZhETF*, **47**, 1303 (1964); **48**, 538 (1956).
5. N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, 1957.
6. C. N. Yang, D. Feldman, *Phys. Rev.*, **79**, 972 (1950).
7. A. D. Sukhanov, *ZhETF*, **43**, 1400 (1962).
8. A. D. Sukhanov, *ZhETF*, **41**, 1915 (1961).
9. C. Schweber, *Introduction to Relativistic Quantum Field Theory*, IL, 1963.
10. W. Weidlich, *Nuovo Cim.*, **30**, 803 (1963).

Note: Figure translations are in progress. See original paper for figures.