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Abstract

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MATHEMATICS

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ON A STRONG MIXING CONDITION FOR STATIONARY GAUSSIAN PROCESSES

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In the present note we study the properties of the spectral density (s.d.) $f(\lambda)$ of a stationary Gaussian process $x(t)$ satisfying the strong mixing condition ([1−4]). In contrast to the purely real methods of note ([3]), the methods of the present paper are connected with the study of the function $\Gamma(z)$ associated with $f(\lambda)$, analytic in the disk (upper half-plane) (see below).

1. Discrete time. In this section we consider processes $x(t)$ with discrete time $t = 0, \pm 1, \ldots$. In view of ([2]), the mixing coefficient is

$$\alpha(\tau) = \sup \left| \int_{-\pi}^{\pi} e^{i\tau\lambda} \varphi(\lambda) f(\lambda)\, d\lambda \right|, \qquad \tau = 0, 1, \ldots, \tag{1}$$

where the supremum is taken over all continuous functions $\varphi(\lambda)$, analytically continuable inside the disk, for which

$$\int_{-\pi}^{\pi} |\varphi(\lambda)| f(\lambda)\, d\lambda = 1.$$

A process $x(t)$ satisfying the strong mixing condition is, of course, regular and, by a well-known theorem of A. N. Kolmogorov,

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda$$

converges.

Put

$$\Gamma(z) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(\lambda) \frac{e^{i\lambda} + z}{e^{i\lambda} - z}\, d\lambda \right\}, \qquad |z| < 1.$$

This function is analytic in the disk $|z| < 1$. Its radial boundary values exist almost everywhere, and almost everywhere $\left|\Gamma(e^{i\lambda})\right| = f(\lambda)$. From Beurling's results ([5]) ($\Gamma(z)$ is an outer function) it follows that the set of functions $\{\varphi(\lambda)\Gamma(e^{i\lambda})\}$, where $\varphi(\lambda)$ runs through all polynomials in nonnegative powers of $e^{i\lambda}$, is dense in the unit sphere of the Hardy space H_1. Therefore, putting

$$f(\lambda) = \Gamma(e^{i\lambda}) \exp\{-i \arg \Gamma(e^{i\lambda})\},$$

we find from (1) that

$$\alpha(\tau) = \sup \left| \int_{-\pi}^{\pi} e^{i\tau\lambda} \varphi(\lambda) \exp\{-i \arg \Gamma(e^{i\lambda})\}\, d\lambda \right|, \tag{2}$$

where the supremum is taken over all elements φ of the unit sphere of H_1.

Put $\ln f(\lambda) = q(\lambda)$. Obviously,

$$\arg \Gamma(e^{i\lambda}) = \tilde{q}(\lambda),$$

where, in general, $\tilde{q}$ denotes the function conjugate (trigonometrically) to q.

Studying the behavior of the integral in (2) for functions $\varphi(\lambda)$ of the form

$$\frac{1}{2\pi N} \left(\frac{e^{iN(\lambda-\mu)} - 1}{e^{i(\lambda-\mu)} - 1} \right)^2,$$

one can prove the following theorem.

Theorem 1. *If $\alpha(\tau) \to 0$, the function $\tilde{q}(\lambda) = \ln f(\lambda)$ has no discontinuities of the first kind, except for jumps whose magnitude is a multiple of 2π.*

This theorem makes it possible to derive in another way the consequence of remark [3] on the order of the zeros of the spectral density $f(\lambda)$, relating in addition the behavior of the zeros of $f(\lambda)$ to the smoothness of $\tilde{q}(\lambda)$.

Theorem 2. *If the function $e^{-i\tilde{q}(\lambda)}$ is continuous, the process $x(t)$ satisfies the strong mixing condition. Moreover,*

$$\alpha(\tau) = O\left(E_{\tau-1}\left(e^{-i\tilde{q}}\right)\right),$$

where $E_\tau(\psi)$ denotes the value of the best approximation of the function $\psi(\lambda)$ by trigonometric polynomials of degree $\leqslant \tau$.

From Theorem 2 follows Corollary 1, which makes it possible to construct processes $x(t)$ with a discontinuous spectral density $f(\lambda)$ that satisfy the strong mixing condition.

Corollary 1. *If at least one of the functions $q(\lambda)$, $\tilde{q}(\lambda)$ is continuous, the process $x(t)$ satisfies the strong mixing condition, and*

$$\alpha(\tau) = O\left(\min\left(E_{\tau-1}(q), E_{\tau-1}(\tilde{q})\right)\right).$$

For example, if

$$f(\lambda) = e^{q(\lambda)} = \exp\left\{\sum_{1}^{\infty} \frac{\cos k\lambda}{k \ln k}\right\},$$

then the process $x(t)$ satisfies the strong mixing condition, since

$$\tilde{q}(\lambda) = \sum_{1}^{k} \frac{\sin k\lambda}{k \ln k}$$

is continuous. Here $f(\lambda)$ has a logarithmic singularity, and

$$\alpha(\tau) = O\left(E_\tau(\tilde{q})\right) = O\left((\ln \tau)^{-1}\right).$$

Theorem 3. *Let $\gamma(\tau)$ be an arbitrary sequence monotonically decreasing to zero. There exists a stationary Gaussian process $x(t)$ for which*

$$\frac{\alpha(\tau)}{\gamma(\tau)} \sim 1.$$

For the proof it is enough to consider the process with spectral density

$$f(\lambda) = A + \sum_{1}^{\infty} [\gamma(k) - \gamma(k+1)] \cos k\lambda,$$

where A is chosen so that $f(\lambda) > m > 0$.

2. Continuous time. In this section we consider processes with continuous time, $-\infty < t < \infty$. Now

$$\alpha(\tau) = \sup \left| \int_{-\infty}^{\infty} e^{i\tau\lambda} \varphi(\lambda) f(\lambda)\, d\lambda \right|, \qquad \tau \geqslant 0, \tag{3}$$

where the supremum is taken over functions $\varphi(\lambda)$ analytically continuable into the upper half-plane and satisfying the condition

$$\int_{-\infty}^{\infty} |\varphi(\lambda)| f(\lambda)\, d\lambda = 1.$$

We first give analogues of Theorems 1 and 2 from [3].

Theorem 4. *Whatever the number a, $0 < a < \infty$, the spectral density $f(\lambda)$ of a process $x(t)$ satisfying the strong mixing condition is representable in the form*

$$f(\lambda) = |P_a(\lambda)|^2 g_a(\lambda),$$

where $P_a(\lambda)$ is a polynomial with real zeros, and the primitive $G_a(\lambda)$ of the function $g_a(\lambda)$ satisfies the following condition: as $h \to 0$, uniformly in $\lambda \in [-a, a]$,

$$|G_a(\lambda + h) + G_a(\lambda - h) - 2G_a(\lambda)| = o\left(G_a(\lambda + h) - G_a(\lambda)\right). \tag{4}$$

From this theorem it follows that Corollaries 1-3 of remark [3] remain valid also for processes $x(t)$ with continuous time.

Theorem 5. *Let the spectral density of a process $x(t)$ satisfying the strong mixing condition be representable in the form*

$$f(\lambda) = |B(\lambda)|^2 g(\lambda), \tag{5}$$

where $B(\lambda)$ is a square-integrable entire function of finite degree, having no nonreal zeros in some strip containing the real axis, and the function $g(\lambda)$ is bounded above and below, $0 < m < g(\lambda) < M$. Then, for the primitive $G(\lambda)$ of the function $g(\lambda)$, condition (4) is satisfied uniformly for all $\lambda \in (-\infty, \infty)$.

Theorem 6. Let the s.d. $f(\lambda)$ be representable in the form (5), where $B(\lambda)$, $g(\lambda)$ are the same as in Theorem 5. If

$$\sum_{n=1}^{\infty} \omega^2(2^{-n}) < \infty,$$

$$\omega(h) = \sup_{\lambda,\, s \le h} \frac{|G(\lambda + s) + G(\lambda - s) - 2G(\lambda)|}{G(\lambda + s) - G(\lambda)},$$

then the process $x(t)$ satisfies the strong mixing condition.

Theorem 5 makes it possible to give examples of processes $x(t)$ whose s.d. is arbitrarily smooth, has regular behavior at infinity ($\asymp (1 + \lambda^2)^p$, and which, nevertheless, do not satisfy the strong mixing condition. Such, for example, are processes with s.d.

$$f(\lambda) = \frac{1 + \sin^2 \lambda^2}{(1 + \lambda^2)^p}, \qquad p > \frac{1}{2}.$$

In order to transfer the results of Sec. 1 to the case of continuous time, put

$$\Gamma(z) = \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\ln f(\lambda)}{1+\lambda^2}\frac{1+z\lambda}{\lambda-z}\,d\lambda\right\}, \qquad \operatorname{Im} z > 0,$$

$$q(\lambda) = \frac{\ln f(\lambda)}{1+\lambda^2}, \qquad Q(\lambda) = (1+\lambda^2)\tilde{q}(\lambda) = \frac{1}{\pi}(1+\lambda^2)\int_{-\infty}^{\infty}\frac{\ln f(x)}{1+x^2}\frac{dx}{x-\lambda}.$$

Arguments similar to those used in Sec. 1 lead to the following theorems.

Theorem 7. If the function $\exp\{iQ(\lambda)\}$ is uniformly continuous, then $\alpha(\tau)\to 0$, and moreover $\alpha(\tau)\le A_\tau(e^{iQ})$, where $A_\tau(\psi)$ is the value of the best approximation of the function $\psi(\lambda)$ by entire functions of finite degree $\le\tau$.

Theorem 8. In order that $\alpha(\tau) = O(\tau^{-r-\beta})$, $r = 0, 1, \ldots;\ 0 < \beta < 1$, it is sufficient that $f(\lambda)$ be representable in the form

$$f(\lambda) = |B(\lambda)|^2 g(\lambda),$$

where $B(\lambda)$ is a square-integrable entire function of finite degree, and $g(\lambda)$ has the following properties:

$$1) \qquad \int_{-\infty}^{\infty}\frac{|\ln g(\lambda)|}{1+\lambda^2}\,d\lambda < \infty;$$

2) the function $\ln g(\lambda)$ has an r-th derivative satisfying a Hölder condition of order β.

Theorem 9. Let the s.d. $f(\lambda)$ be representable in the form

$$f(\lambda) = |B(\lambda)|^2 g(\lambda),$$

where:

1) $B(\lambda)$ is a bounded entire function of finite degree;
2) for some real p, $-\infty < p < \infty$, $|\lambda|^p g(\lambda) \asymp 1$, $\lambda\to\infty$;
3) the function $\ln g(\lambda)$ is uniformly continuous, or, equivalently,

$$\sup_{\lambda,\, t\le h}\frac{|g(\lambda+t)-g(\lambda)|}{g(\lambda)} = \omega(h)\downarrow 0, \quad h\to 0.$$

Then the process $x(t)$ satisfies the strong mixing condition, and moreover $a(\tau) = O(\omega(1/\tau))$.

3. Generalized processes. Let now $x(\varphi)$ be a generalized stationary Gaussian process in the sense of K. Itô–I. M. Gelfand (see [6]). Naturally, we shall say that it satisfies the strong mixing condition if

$$|Ex(\varphi)x(\psi)| \leq \alpha(\tau)\left(E|x(\varphi)|^2 E|x(\psi)|^2\right)^{1/2}, \quad \alpha(\tau) \downarrow 0,$$

where the supports of the functions φ and ψ are located respectively in $(-\infty, 0]$, $[\tau, \infty)$, $\tau \geq 0$. The mixing coefficient $\alpha(\tau)$ in this case too is defined by equality (3), where now $f(\lambda)$ is the s.d. of the process $x(\varphi)$.

Lemma 1. *The mixing coefficients of generalized stationary processes with s.d. $f(\lambda)$ and $f(\lambda)(1+\lambda^2)^p$, $-\infty < p < \infty$, differ by a quantity of order $O(e^{-c\tau})$, $c = c(p) > 0$.*

It follows from this lemma that all the theorems of Sec. 2 remain valid also for generalized processes.

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REFERENCES

[1] M. Rosenblatt, Proc. Nat. Acad. Sci. USA, **42**, 43 (1956).
[2] A. N. Kolmogorov, Yu. A. Rozanov, Theory of Probability and Its Applications, **5**, iss. 2 (1960).
[3] I. A. Ibragimov, DAN, **147**, No. 6 (1962).
[4] Yu. A. Rozanov, *Stationary Random Processes*, Moscow, 1963.
[5] A. Beurling, Acta Math., **81**, 239 (1949).
[6] I. M. Gelfand, N. Ya. Vilenkin, *Some Applications of Harmonic Analysis*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.