EXTREMAL SPACES

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Abstract

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EXTREMAL SPACES

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In view of the primary role played in modern applied mathematics by extremization operations (finding maxima and minima), the question arises of an independent study of these operations from a certain general point of view. It is true that in most applications these operations are applied not to the elements of a space themselves, but to certain functionals of them. In other words, extremal operations occur together, say, with the operation of addition. However, one can list a number of situations that admit an adequate description in terms of extremal operations alone (and, possibly, the operation of multiplication). The study of extremization operations from various standpoints has already been undertaken by a number of authors. Thus, Bellman and Karush, in their papers on the maximum-transform (^{1}), considered this question from a purely analytic point of view; in the paper of N. N. Vorob' ev (^{2}) an approach in the spirit of classical matrix algebra is presented. Here we shall follow a somewhat different path. We introduce the concept of an extremal space as a set in which the operations of taking the maximum and minimum of any two of its elements and the operation of multiplication of elements by "scalars" from a certain set of a special kind are defined. Such objects occupy an intermediate position between structures (^{3}) and linear spaces, possessing a number of features of both. Here we shall present some initial facts of the theory that arises under this approach.

Let $K = \{\xi, \eta, ...\}$ be a certain set on which are given: 1) multiplication, with respect to which K is an abelian group; 2) the operations $\xi \vee \eta$ (maximization) and $\xi \wedge \eta$ (minimization), with respect to which K is a structure with the usual axioms (^{3}). We shall provisionally call K an "extremal field." An extremal space is a set $E = \{x, y, ...\}$ on which the operations $x \vee y$ and $x \wedge y$ are defined (with respect to which E is also a structure) as well as the operation of multiplication ξx, $\xi \in K$, $x \in E$.

The following axioms are assumed to hold:

1°. $\xi(x * y) = \xi x * \xi y$.

2°. $(\xi * \eta)x = \xi x * \eta x$.

3°. $\xi(\eta x) = (\xi\eta)x$ (here $*$ denotes any of the symbols $\vee, \wedge$).

In E one can introduce a partial order relation $\geqq$, setting $x \geqq y$ if $x \vee y = x$ (the relation $\leqq$ is defined dually). Further, in E there is a least element 0, defined by the axiom

4°. $0 \vee x = x$.

An extremal form in the variables $\xi_1, \dots, \xi_p \in K,\ x_1, \dots, x_q \in E$ is defined to be: 1) any of the x_k; 2) any product $\xi_i x_k$; 3) any result of applying to these forms the operations $\vee$ and $\wedge$, and so on by induction. In various technical constructions of this theory the following principle proves useful.

Theorem 1. *An extremal form is an isotone function of all its arguments (of any subset of arguments).*
(3)

Elements $x, y \in E$ are called disjoint if $x \wedge y = 0$. Defining in the generally accepted way an isomorphism of two spaces over the same extremal field, we immediately arrive at the following theorem.

Theorem 2. *An isomorphism carries zero elements into one another, preserves the partial order and disjointness of elements.*

An important example of an extremal space is the nonnegative orthant R_n^+, in which multiplication by nonnegative scalars is understood in the usual way, while the maximization and minimization of two vectors are taken componentwise.

An extremal space is called distributive if the following axiom holds in it:

5^0.

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A system of elements $\{x_i\}$ will be called, as in the theory of structures, independent if

$$\left(\bigvee_{i \neq k} x_i \right) \wedge x_k = 0.$$

It is easy to establish that in a distributive space independence is equivalent to pairwise disjointness of the elements. Furthermore, independence is invariant with respect to multiplication by elements of K. We shall say that an element y is representable in the form of a linear combination of elements x_i, if the x_i are independent and there exist coefficients $\lambda_i \in K$ such that

$$y = \bigvee_i \lambda_i x_i.$$

By the length of a combination we shall mean the number of independent x_i entering into it. If in a linear combination some x_i, in turn, are represented as linear combinations, then the result of the "superposition" of such combinations will be called a continuation of the initial linear combination.

Theorem 3. *The coefficients of a linear combination are determined uniquely.*

Theorem 4. *In order that a distributive extremal space E be isomorphic to R_n^+, it is necessary and sufficient that every element of E be decomposable into a linear combination, every continuation of which has length not exceeding n.*

An Archimedean component of an extremal space will be such a subset $\widetilde{E}$ of it on which the following axiom holds:

6^0. For any $x, y \in \widetilde{E}$ there exists a $\lambda \in K$ such that $\lambda x \geq y$.

By an extremal metric $\rho(x, y)$ we shall mean a mapping of $E \times E$ into K satisfying the axioms:

7^0. $\rho(x, y) \geq 1$; $\rho(x, y) = 1$ if and only if $x = y$.

8^0. $\rho(x, y) = \rho(y, x)$.

9^0. $\rho(x, y) \leq \rho(x, z)\rho(z, y)$.

Theorem 5. *Every Archimedean component is extremally metrizable.*

Theorem 6. *An extremal space decomposes into the set-theoretic union of its pairwise nonintersecting Archimedean components.*

We emphasize that for x and y belonging to different Archimedean components, $\rho(x, y)$ is not defined.

An extremal functional f in E will be called a mapping of E into K satisfying the axioms:

10^0.

$$f(x \vee y) = f(x) \vee f(y).$$

11^0.

$$f(\lambda x) = \lambda f(x).$$

A functional p is called homogeneous if it satisfies only axiom 11^0; a homogeneous functional is called convex if axiom 10^0 holds for it with the sign $\leq$. We also introduce a scalar product $[x, y]$, possessing the natural properties:

12^0.

$$[x, x] = 0$$

if and only if $x = 0$ (here 0 is the minimal element of the structure K).

13°. $[x, y] = [y, x]$.

14°. $[\lambda x, y] = \lambda[x, y]$.

15°. $[x \vee y, z] = [x, z] \vee [y, z]$.

16°. $[x \wedge y, z] = [x, z] \wedge [y, z]$.

If the extremal field is taken to be $K = R_1^+$, we can also introduce the extremal norm of an element x, understanding by it

$$|||x||| = \sqrt{[x, x]}.$$

The following analogue of the well-known Riesz theorem holds (4).

Theorem 7. *The scalar product $[x, a]$ is an extremal functional; conversely, every such functional f in the space R_n^+ is representable in the form $[x, a]$, where a is a fixed element of R_n^+.*

Hence it follows that

Theorem 8. *The upper space R_n^+ (i.e., a space in which only the "upper" algebraic operation $\vee$ is given) and the space of extremal functionals over it are isomorphic.*

Theorem 9. *In distributive spaces the notions of disjointness, independence, and orthogonality in the sense of the scalar product $[x, a]$ are equivalent.*

Theorem 10. *Let, in the space R_n^+, the vector x be such that $x \geqq 1 = (1, \dots, 1)$. Then $|||x||| = \tilde{\rho}(x, 1)$.*

By a hyperplane in $\tilde{E}$ we shall mean the set

$$H_f = \{x \mid f(x) = \gamma\},$$

where γ is a fixed element of K. A set T is called extremally convex (2) (from above) if from $x, y \in T$, $\lambda, \mu \in K$, $\lambda \vee \mu = 1$, it follows that $\lambda x \vee \mu y \in T$. It is easy to establish that the set $T_\alpha = \{x \mid p(x) \leqq \alpha\}$, where p is an extremal or convex functional, is extremally convex and its boundary $\operatorname{Fr} T_\alpha$ consists of the points for which $p(x) = \alpha$, and only of them. A set C is called an (upper) extremal cone (2) if from $x, y \in C$ it follows that $\lambda x \vee \mu y \in C$ for any $\lambda, \mu \in K$. We note that in the present theory cones play a role analogous to that of subspaces of a linear space.

Theorem 11. *Let C be an extremal cone lying in a certain Archimedean component $\tilde{E}$, $x_0 \in \tilde{E} \setminus C$, $C' = C \vee \lambda x_0$, $\lambda \in K$. Let an extremal functional f be given on C, and on $\tilde{E}$ a convex functional p, with $f(x) \leqq p(x)$, $x \in C$. Then f can be extended to an extremal functional f', defined on C', with $f'(x) \leqq p(x)$, $x \in C'$.*

With the help of this theorem one establishes the following variant of the Hahn-Banach theorem (4), from which, as in the linear case, various geometric consequences are easily derived.

Theorem 12. *Let $C \subset \tilde{E}$ be an extremal cone. Let an extremal functional f be given on C, and on the whole Archimedean component $\tilde{E}$ a convex functional p satisfying the condition $f(x) \leqq p(x)$, $x \in C$. Then f can be extended to an extremal functional F, defined on $\tilde{E}$ and satisfying the condition $F(x) \leqq p(x)$, $x \in \tilde{E}$.*

Theorem 13. *Let p be a convex functional on $\tilde{E}$. Then through any boundary point of the extremally convex set T_α defined above one can draw a supporting hyperplane to T_α.*

Theorem 14. *Let p be a convex functional on some Archimedean component $\tilde{E}$, $T_\alpha \subset \tilde{E}$, $x' \in \tilde{E} \setminus (T_\alpha \cup \operatorname{Fr} T_\alpha)$. Then there exists a supporting hyperplane separating x' from T_α.*

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Note: Figure translations are in progress. See original paper for figures.

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