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Abstract

Full Text

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MATHEMATICS

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ON GRAPHS OF A SPECIAL TYPE

(Presented by Academician S. A. Lebedev on 14 XII 1964)

In solving certain problems connected with the automation of the design of computing units and systems, it has proved useful to investigate the properties of graphs of a special type.

Consider a set of points $\mathfrak{A}$ situated at the nodes of an infinite rectangular lattice. Two points of $\mathfrak{A}$ shall be called **neighboring** if they are the nearest points along one horizontal, vertical, or diagonal line. A graph G shall be called a **scheme** if the vertices of the graph belong to $\mathfrak{A}$, and edges join vertices situated at neighboring points (Fig. 1). Not every graph can be represented in the form of a scheme. Graphs admitting such a representation will be called **regular graphs** (r.g.).

From this definition it immediately follows:
a) to each scheme there corresponds a graph (a regular one), and indeed only one;
b) to each r.g. there corresponds at least one scheme.

A scheme in which all neighboring vertices are joined by edges is naturally called a **complete scheme**. A complete scheme having as its vertices all points of the set $\mathfrak{A}$ shall be called a **limiting complete scheme**, and the graph corresponding to it a **limiting regular graph**. Obviously, any r.g. is a subgraph of a limiting r.g.; therefore consideration of the latter gives us certain upper estimates for the properties of regular graphs.

Fig. 1. Scheme of a 2-connected r.g.

1. *The degree of any vertex of an r.g. is* $r \leqslant 8$ *(obvious).*
2. *In an r.g. there are no loops and no cycles of length two (obvious).*

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Fig. 2

Figure 2: Fig. 2

3. *The cyclomatic number of an r.g. is* $\nu < 3n + p$, *where* n *is the number of vertices, and* p *is the number of connected components.*

Indeed, the degree of any vertex of an r.g. is $r \leqslant 8$; consequently, the number of edges of an r.g. is $m < 4n$. Hence

$$\nu = m - n + p < 4n - n + p = 3n + p.$$

4. *The chromatic number of an r.g. is* $\gamma \leqslant 4$.

Consider a limiting complete scheme. Let us partition the set of its vertices $\mathfrak{A}$ into four disjoint sets A_1, A_2, A_3, and A_4 in the following way. Take an arbitrary vertex a_0. To the set A_1 assign all vertices lying at the nodes of the rectangular lattice with doubled step and including the vertex a_0. To the set A_2 assign all vertices lying at the nodes of the rectangular lattice with doubled step and including the vertices neighboring a_0 along the diagonals. To the set A_3 assign all vertices lying at the nodes of the rectangular lattice with doubled step and including the vertices neighboring a_0 along the vertical. Finally, to the set A_4 assign all vertices lying at the nodes of the rectangular lattice with

by a doubled step and including the vertices adjacent to a_0 horizontally. It is easy to see that

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \mathfrak{A}, \qquad A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset.$$

It follows from the method of construction that no two vertices of any of the sets A_1, A_2, A_3, and A_4 are adjacent. Consequently, we have colored the vertices of the limiting complete scheme, or of the limiting planar graph, in four colors so that no two vertices are colored the same color. On the other hand, the limiting planar graph contains, as subgraphs, complete graphs on four vertices. This means that it is impossible to color the vertices of the limiting planar graph in fewer than four colors. Hence the chromatic number of the limiting planar graph is equal to 4 (see Fig. 2). From this property 4 follows immediately, since any planar graph is a subgraph of the limiting planar graph.

Fig. 2. $\gamma = 4$, $\beta = 8$. a—vertices of the 1st color; —vertices of the 2nd color; —vertices of the 3rd color; —vertices of the 4th color

5. The chromatic class of a planar graph is $\beta \leqslant 8$. Again consider the limiting complete scheme. Color its edges in the following way. Take an arbitrary vertex a_0. Color the edges incident with it, beginning with an arbitrary edge, with the colors $1, 2, 3, 4, 5, 6, 7, 8$ in the order in which they occur if one goes around the vertex clockwise. Next we pass to a vertex adjacent to a_0. If this adjacency is along a diagonal, then, taking into account the edges already colored, we color the edges incident with the new vertex with the colors $1, 2, 3, 4, 5, 6, 7, 8$, preserving the same order when going around the new vertex clockwise. If this adjacency is horizontal or vertical, then we change the order of the colors to the opposite one. Next we pass to a vertex adjacent to the preceding one and color, according to the indicated rule, the edges incident with it. It is not difficult to verify that, by repeating this process, one can color all the edges of the limiting complete scheme or of the limiting planar graph. In doing so, no edges incident with one vertex are colored the same color. Knowing that the degrees of the vertices of the limiting planar graph are equal to 8, we may assert that it is impossible to color the edges in a smaller number of colors. Consequently, the chromatic class of the limiting planar graph is equal to 8 (see Fig. 2). From this property 5 follows at once, since any planar graph is a subgraph of the limiting planar graph.

6. The radius of a regular graph with m edges and n vertices satisfies the relations:

$$\rho \geqslant \frac{\sqrt{n}-1}{2}; \qquad \rho \geqslant \frac{\sqrt{4m+1}-1}{8}.$$

Consider a complete scheme of $(2\rho+1)^2$ vertices lying inside a square of side $(2\rho+1)$. It is obvious that from the vertex lying at the center of the square there exist paths to all the other vertices of length not exceeding ρ edges. On the other hand, no vertex outside the square can be joined to the center by a path of length not exceeding ρ. Hence the planar graph $G^{\square}$ corresponding to the given complete scheme contains all planar graphs of radius not greater than ρ as its subgraphs. The graph $G^{\square}$ contains $N = (2\rho+1)^2$ vertices and $4\rho(4\rho+1) = M$ edges. Then for any planar graph with radius not greater than ρ,

$$n \leqslant N = (2\rho+1)^2; \qquad m \leqslant M = 4\rho(4\rho+1),$$

whence property 6 is easily obtained. For an arbitrary graph the following lower estimate for the radius (1) is known:

$$\rho \geq \log(np - n + 1)/\log p - 1,$$

where p is the maximum degree of a vertex of the graph. For a c.g. this leads to

$$\rho \geq \log(7n+1)/\log 8 - 1,$$

which is somewhat worse than the result obtained by us. It is easy to construct an example showing that, for a fixed number of vertices n, the complete scheme

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Fig. 3 and Fig. 4

Figure 3: Fig. 3 and Fig. 4

of minimum radius need not contain the greatest number of edges (Figs. 3I and 3II).

Fig. 3. I—$n = 18, \quad m = 46, \quad \rho = 2;$
II—$n = 18, \quad m = 49, \quad \rho = 3; \quad III$—$n =$
$= 7, \quad m = 10, \quad \gamma = 2, \quad \beta = 5$

Fig. 4

All six of the listed properties of c.g.' s are not sufficient for an arbitrary graph to be a c.g. It is not difficult to verify that the graph shown in Fig. 3III is not a c.g., although it satisfies all the conditions.

For the proof of property 6 we considered a graph that is interesting in that it contains, as subgraphs, all c.g.' s with radius not exceeding ρ. Number all vertices of the corresponding complete scheme in the natural way: from left to right, starting with the first horizontal, and so on to the last. It is clear that the incidence matrix of the graph will have the following special form (Fig. 4): the nonzero elements will be located on the first (2ρ)-th, $(2\rho+1)$-th, and $[2(\rho+1)]$-th diagonals, if counted from the main diagonal in both directions.

Hence the following property may be formulated:

7. *In a c.s. there exists a numbering of the vertices* $1, 2, \dots, (2\rho+1)^2 \geq n$, *for which the incidence matrix has the form shown in Fig. 4.*

What is meant is that another $[(2\rho+1)^2 - n]$ isolated vertices are added to the n vertices of the graph, completing the graph scheme to a square, and the indicated diagonals are not completely filled with ones. In order to construct exactly the incidence matrix of the given graph, it is necessary to set equal to zero those elements lying on the indicated diagonals that correspond to edges joining the left and right sides of the square. These will be the elements a_{ij}, where $i \neq j$,

$$i = k_1(2\rho+1), \quad k_1 = 1, 2, \dots, (2\rho+1),$$

$$j = k_2(2\rho+1)+1, \quad k_2 = 1, 2, \dots, 2\rho, \qquad |k_1 - k_2| \leq 1.$$

Denote the resulting matrix by A^*. Then

In order for a graph to be an s.g., it is necessary and sufficient that the incidence matrix of the graph, augmented by $[(2\rho+1)^2 - n]$ *zero rows and columns, can be brought to the form* A^* *by a permutation of elements.*

In connection with important technical applications of s.g.' s, the following problems are of considerable interest:

A. Given an s.g., for example by its incidence matrix, find for it an algorithm for constructing a circuit.

B. Given an arbitrary graph G, find an algorithm for constructing a circuit for a subgraph of G with the same number of vertices and the maximum number of edges.

In solving these problems, the realizability of the algorithms on existing digital computers is envisaged.

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Note: Figure translations are in progress. See original paper for figures.

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