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Abstract

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A Priori Estimates for Solutions of Linear Second-Order Equations of Elliptic and Parabolic Types

(Presented by Academician A. D. Aleksandrov on 17 X 1964)

In this paper integral and pointwise estimates are formulated, obtained by the author for classes of functions satisfying inequalities of a special form and including solutions of various second-order differential equations. These results are used in obtaining a priori estimates for solutions of linear second-order equations of elliptic and parabolic types. For linear equations of parabolic type an existence and uniqueness theorem is established for a certain class of generalized solutions. Results close to certain special cases 1) and 4) of Theorem 3 are contained in ([6]). Theorem 3, case 4), and Theorem 4 were proved in ([7]), but for a narrower class of generalized solutions. In the paper the methods created in ([1,2]) are developed.

Let Ω denote a certain domain in the space E_n, and let Q denote the cylinder $\Omega \times [t_1, t_2]$. By $K_\rho = K_\rho(x_0)$ we shall denote the ball

$$|x - x_0| \le \rho, \quad |x - x_0| = \left(\sum_{i=1}^{n} (x_i - x_{0i})^2 \right)^{1/2},$$

and by $Q_{\rho,\tau} = Q_{\rho,\tau}(x_0, t_0)$ the cylinder

$$K_\rho(x_0) \times [t_0 - \tau, t_0].$$

If a certain set D and a certain function $v(P)$, defined and measurable in D, are given, then by $\{D, v > s\}$ is denoted the set of those points P in D where $v(P) > s$. We shall also denote by $\|v\|_{m,D}$ the expression

$$\|v\|_{m,D} = \left(\operatorname{mes}^{-1} D \cdot \int_D |v|^m dP \right)^{1/m}.$$

Let, as usual, the function $v_+(P)$ denote the positive part of the function $v(P)$, i.e.

$$v_+(P) = \frac{1}{2}(|v(P)| + v(P)).$$

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By $W_m^1(\Omega)$ and $W_m^1(Q)$ are denoted the well-known spaces of S. L. Sobolev. We introduce the spaces $V_m^{1,0}(Q)$ and $\overset{\circ}{V}{}_m^{1,0}(Q)$. Let the norm $\|u\|_{V_m^{1,0}(Q)}$ be defined for all functions $u(x,t)$ continuously differentiable in the cylinder $\overline{Q}$ by

$$\|u\|_{V_m^{1,0}(Q)} = \max_{t\in[t_1,t_2]} \left(\int_\Omega |u|^m dx\right)^{1/m} + \left(\iint_Q |u_x|^m dx\, dt\right)^{1/m}. \tag{1}$$

By the space $V_m^{1,0}(Q)$ $[\overset{\circ}{V}{}_m^{1,0}(Q)]$ we shall mean the closure, with respect to the norm (1), of the set of all smooth functions in $\overline{Q}$ (the set of all smooth functions equal to zero near the lateral surface S_{lat} of the cylinder Q). We note that functions from $V_m^{1,0}(Q)$ have generalized derivatives $u_{x_i} \in L_m(Q)$ and are continuous in t in $[t_1, t_2]$ as elements of $L_m(\Omega)$.

O. A. Ladyzhenskaya and N. N. Ural' tseva, in connection with the study of properties of solutions of second-order elliptic equations in (^{2e}) (see also ($^{2a-2d}$)), considered the class of functions $v(x) \in W_m^1(K_r)$, $1 < m \le n$, satisfying, for all $k \ge k'$, $\sigma \in (0,1)$, $\rho \in [r/2, r]$, the inequalities

of the form

$$\int_{\{K_{\rho-\sigma\rho},\, v>k\}} |v_x|^m\, dx \le \gamma_1(\sigma\rho)^{-m} \int_{\{K_\rho,\, v>k\}} (v-k)^m\, dx + \gamma_2 k^\alpha \rho^{-n\varepsilon} \operatorname{mes}^{1-m/n+\varepsilon}\{K_\rho,\, v>k\}, \tag{2}$$

where $k' \ge 0$, $\gamma_1 \ge 0$, $\gamma_2 \ge 0$, $0 \le \alpha < m + m\varepsilon$, and $\varepsilon > 0$. For functions of such a class, an estimate of the maximum of the modulus has been obtained. Here we consider inequalities (2) also for nonpositive values of ε, and obtain estimates in weaker norms than $\|u\| = \max|u|$.

Lemma 1. Let the numbers $r > 0$, $1 < m \le n$, and the numbers $\varepsilon, \alpha \ge 0$, $\gamma_1 \ge 0$, $\gamma_2 \ge 0$, $k' \ge 0$ be fixed. Suppose that the function $v(x) \in W_m^1(K_r)$ and, for all $k \ge k'$, $\sigma \in (0,1)$, $\rho \in [r/2, r]$, satisfies inequality (2).

1) Let $-(n-m)/m < \varepsilon < 0$, $0 \le \alpha \le m$. Then

$$r^{-n} \operatorname{mes}\{K_{r/2},\, v > s\} \le c_1 s^{-(m-\alpha)/|\varepsilon|} \left(\|v_+\|_{m,K_r} + 1\right)^{(m-\alpha)/|\varepsilon|};$$

$$s \ge s_0 = c_0^{(1)}(\|v_+\|_{m,K_r} + 1). \tag{3}$$

2) Let $\varepsilon = 0$, $0 \le \alpha \le m$. There exists a number $\delta > 0$, determined only by the numbers n, m, such that, under the condition $\gamma_2 \le \delta e^{-p}$, and for any $p > 1$, the inequality

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$$r^{-n}\,\mathrm{mes}\{K_{r/2},\ v>s\}\le c_2 s^{-p}\big(\|v_+\|_{m,K_r}+1\big)^p;$$

$$s\ge s_0=c_0^{(2)}(\|v_+\|_{m,K_r}+1). \tag{4}$$

holds.

3) Let $\varepsilon=0$, $0\le\alpha<m$. Then

$$r^{-n}\,\mathrm{mes}\{K_{r/2},\ v>s\}\le c_3\exp\left\{-c_4\big(\|v_+\|_{m,K_r}+1\big)^{-(m-\alpha)/m}s^{(m-\alpha)/m}\right\},$$

$$s\ge s_0=c_0^{(3)}(\|v_+\|_{m,K_r}+1). \tag{5}$$

In (3)–(5), the constants $c_0^{(i)}$ and c_j $(j\ne 2)$ are determined only by the numbers $n,m,\alpha,\varepsilon,\gamma_1,\gamma_2,k'$, while the constant c_2 is determined by the same numbers and by the number p.

In the study of parabolic equations, in (2^e) (see also ($2a$-$2d$)) there was introduced and studied a class of functions $v(x,t)$ from $W_m^1(Q_{r,r^m})$, $1<m\le n$ (let $Q_{r,r^m}=Q_{r,r^m}(0,0)$), satisfying, for all $k\ge k'$, $\sigma\in(0,1)$, $\rho\in[r/2,r]$, and for almost all $t\in[-r^m,0]$, inequalities of the form

$$\frac{\partial}{\partial t}\int_{\{K_{\rho-\sigma\rho},\,v(t)>k\}}(v-k)^m\,dx+\nu\int_{\{K_{\rho-\sigma\rho},\,v(t)>k\}}|v_x|^m\,dx\le$$

$$\le\gamma_1(\sigma\rho)^{-m}\int_{\{K_\rho,\,v(t)>k\}}(v-k)^m\,dx+\gamma_2k^\alpha\rho^{-n\varepsilon}\,\mathrm{mes}^{1-m/n+\varepsilon}\{K_\rho,\ v(t)>k\},$$

where $\nu>0$, $\gamma_1\ge 0$, $\gamma_2\ge 0$, $k'>0$, $0\le\alpha<m+m\varepsilon$, $\varepsilon>0$.

For the study of parabolic equations whose coefficients are subject to other conditions, we introduce and study (see ([8])) the class of functions $v(x,t)\in V_m^{1,0}(Q_{r,r^m})$, satisfying, for all $k\ge k'$, $\sigma\in(0,1)$, $\theta\in(0,1)$, $\rho\in[r/2,r]$, $\tau\in[r^m/2,r^m]$, inequalities of the form

$$\max_{t\in[-\tau,0]}\int_{\{K_{\rho-\sigma\rho},\,v(t)>k\}}(v-k)^m\,dx+\iint_{\{Q_{\rho-\sigma\rho,\tau-\theta\tau},\,v>k\}}|v_x|^m\,dx\,dt\le \tag{6}$$

$$\le\gamma_1\left[(\sigma\rho)^{-m}+(\theta\tau)^{-1}\right]\iint_{\{Q_{\rho,\tau},\,v>k\}}(v-k)^m\,dx\,dt+$$

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$$+\gamma_2 k^\alpha \rho^{-n\varepsilon}\tau^{-\varepsilon}\,\mathrm{mes}^{1-m/(m+n)+\varepsilon}\{Q_{\rho,\tau},\ v>k\}.$$

Lemma 2. Let the numbers $r>0$, $m\ge 1-n/2+\sqrt{n^2+4}/2$, and the numbers $\varepsilon,\alpha\ge 0$, $\gamma_1\ge 0$, $\gamma_2\ge 0$, $k'\ge 0$ be fixed. Suppose that the function

$v(x,t)\in V_m^{1,0}(Q_r,r^m)$, and for all $k\ge k'$, $\sigma\in(0,1)$, $\theta\in(0,1)$, $\rho\in[r/2,r]$ and $\tau\in[r^m/2,r^m]$ satisfies inequality (6).

1) Let $-n/(m+n)<\varepsilon<0,\ 0\le\alpha\le m$. Then

$$r^{-(m+n)}\,\mathrm{mes}\{Q_{r/2,r^m/2},\ v>s\}\le c_1 s^{-(m-\alpha)/|\varepsilon|}\left(\|v_+\|_{m,Q_{r,r^m}}+1\right)^{(m-\alpha)/|\varepsilon|};$$

$$s\ge s_0=c_0^{(1)}\left(\|v_+\|_{m,Q_{r,r^m}}+1\right). \tag{7}$$

2) Let $\varepsilon=0,\ 0\le\alpha\le m$. There exists a number $\delta>0$, determined by the numbers n,m, such that under the condition $\gamma_2\le\delta e^{-p}$, for any $p>1$ the inequality

$$r^{-(m+n)}\,\mathrm{mes}\{Q_{r/2,r^m/2},\ v>s\}\le c_2 s^{-p}\left(\|v_+\|_{m,Q_{r,r^m}}+1\right)^p;$$

$$s\ge s_0=c_\theta^{(2)}\left(\|v_+\|_{m,Q_{r,r^m}}+1\right). \tag{8}$$

holds.

3) Let $\varepsilon=0,\ 0\le\alpha<m$. Then

$$r^{-(m+n)}\,\mathrm{mes}\{Q_{r/2,r^m/2},\ v>s\}\le c_3\exp\left\{-c_4\left(\|v_+\|_{m,Q_{r,r^m}}+1\right)^{-(m-\alpha)/m}s^{(m-\alpha)/m}\right\},$$

$$s\ge s_0=c_0^{(3)}\left(\|v_+\|_{m,Q_{r,r^m}}+1\right). \tag{9}$$

4) Let $\varepsilon>0,\ 0\le\alpha\le m$. Then

$$\operatorname*{vrai\,max}_{\{Q_{r/2,r^m/2},\,v>0\}} v\le c_5\left(\|v_+\|_{m,Q_{r,r^m}}+1\right). \tag{10}$$

In inequalities (7)–(10) the constants $c_0^{(i)}, c_j (j\ne 2)$ depend on $n,m,\varepsilon,\alpha,\gamma_1,\gamma_2,k'$, and the constant c_2 on $n,m,\varepsilon,\alpha,\gamma_1,\gamma_2,k',p$.

2. Consider in the cylinder $Q_T=\Omega\times[0,T]$ the equation

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$$\frac{\partial u}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}\frac{\partial u}{\partial x_j}+a_iu+f_i\right)+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu+g=0,\qquad n\geq 1,\ (11)$$

where

$$\nu\sum_{i=1}^{n}\xi_i^2\leq\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\leq\mu\sum_{i=1}^{n}\xi_i^2,\qquad \nu,\mu>0;\qquad a_i\in L_{n+2}(Q_T);$$

$$b_i\in L_{n+2}(Q_T);\qquad c\in L_{(n+2)/2}(Q_T);\qquad f_i\in L_2(Q_T);\qquad g\in L_{2(n+2)/(n+4)}(Q_T).$$

A generalized solution (g.s.) of equation (11) of the class $V_2^{1,0}(Q_T)$ ($\dot V_2^{1,0}(Q_T)$) is a function $u(x,t)\in V_2^{1,0}(Q_T)$ ($\dot V_2^{1,0}(Q_T)$) satisfying, for all t_1 and t_2 from the interval $[0,T]$ and $\Phi(x,t)\in\dot W_2^1(\Omega\times[t_1,t_2])$, the equality

$$\int_\Omega u\Phi\,dx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_\Omega\left[-u\Phi_t+(a_{ij}u_{x_j}+a_iu+f_i)\Phi_{x_i}+(b_iu_{x_i}+cu+g)\Phi\right]dx\,dt=0.$$

Theorem 1. Every g.s. of equation (11) of the class $\dot V_2^{1,0}(Q)$ satisfies the inequality

$$\|u(x,t)\|_{V_2^{1,0}(Q_T)}\leq c\left[\sum_{i=1}^{n}\|f_i(x,t)\|_{L_2(Q_T)}+\|g\|_{L_{2(n+2)/(n+4)}(Q_T)}+\|u(x,0)\|_{L_2(\Omega)}\right],$$

where $c=c(\nu^{-1},T/\tau)$; τ is determined from the equality

$$\sup_{\substack{0\leq t_1\leq T-\tau\\ i=1,\dots,n}}\|a_i^2+b_i^2+c\|_{L_{(n+2)/2}(\Omega\times[t_1,t_1+\tau])}=\nu\min(1,\nu)\delta,\qquad \delta=\delta(n)>0.$$

Theorem 2. For every $\varphi(x)\in L_2(\Omega)$ there exists a unique g.s. of equation (11) of the class $\dot V_2^{1,0}(Q_T)$, equal to $\varphi(x)$ for $t=0$.

If the g.s. $u(x,t)$ of equation (11) of the class $V_2^{1,0}(Q_T)$ satisfies the condition $\|u(x,t+h)-u(x,t)\|_{L_2(Q_T)}=o(h^{1/2})$, then it will be called a g.s. of the class $V_2^{1,1/2}(Q_T)$.

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Theorem 3. Let $u(x,t)$ be an arbitrary generalized solution of equation (11) from the class $V_2^{1,1/2}(Q_T)$. There exists a quantity d (depending only on n and ν in cases 1), 3), and 4) singled out below, and only on n, ν, p in case 2)) such that, if the constant $r_0 > 0$ is determined from the equality

$$\sup_{\substack{Q_{r_0,r_0^2} \subset Q_T \\ i=1,\dots,n}} \left\| a_i^2 + b_i^2 + c \right\|_{L_{(n+2)/2}(Q_{r_0,r_0^2})} = d,$$

then in every cylinder Q_{r,r^2}, $r \le r_0$, the following estimates hold for the solution u:

1) Let $a_i \in L_{n+2}(Q_T)$, $b_i \in L_{n+2}(Q_T)$, $c \in L_{(n+2)/2}(Q_T)$, $f_i \in L_q(Q_T)$, $2 < q < n+2$, $g \in L_p(Q_T)$, $2(n+2)/(n+4) < p < (n+2)/2$. Then, for any $\sigma > 0$,

$$\|u\|_{p_*-\sigma, Q_{r/2,r^2/2}} \le c_1(\|u\|_{2,Q_{r,r^2}} + 1); \qquad p_* = \min\left[\frac{q(n+2)}{n+2-q}, \frac{p(n+2)}{n+2-2p}\right].$$

2) Let $a_i \in L_{n+2}(Q_T)$, $b_i \in L_{n+2}(Q_T)$, $c \in L_{(n+2)/2}(Q_T)$, $f_i \in L_{n+2}(Q_T)$, $g \in L_{(n+2)/2}(Q_T)$. Then, for any $p > 1$,

$$\|u\|_{p,Q_{r/2,r^2/2}} \le c_2(\|u\|_{2,Q_{r,r^2}} + 1).$$

3) Let $a_i \in L_q(Q_T)$, $b_i \in L_{n+2}(Q_T)$, $c \in L_{q/2}(Q_T)$, $f_i \in L_{n+2}(Q_T)$, $g \in L_{(n+2)/2}(Q_T)$, $q > n+2$. Then

$$r^{-(n+2)} \iint_{Q_{r/2,r^2/2}} \exp\left\{c_3(\|u\|_{2,Q_{r,r^2}} + 1)^{-1}|u(x,t)|\right\} dx\,dt \le c_4.$$

4) Let $a_i \in L_q(Q_T)$, $b_i \in L_{n+2}(Q_T)$, $c \in L_{q/2}(Q_T)$, $f_i \in L_q(Q_T)$, $g \in L_{q/2}(Q_T)$, $q > n+2$. Then

$$\operatorname{vrai\,max}_{Q_{r/2,r^2/2}} |u| \le c_5(\|u\|_{2,Q_{r,r^2}} + 1).$$

In the inequalities given above, the constants c_i are determined by the numbers n, ν, μ and by the norms $\|\cdot\|_{L_p(Q_{r,r^2})}$ of the coefficients a_i, c, f_i and g with those exponents p which are indicated in the hypotheses of these inequalities. In addition, the constant c_1 also depends on σ and r_0, and the constant c_2 on p.

Theorem 4. Under the assumptions of Theorem 3, 4), the estimate

$$|u|_{C_{\alpha,\alpha/2}(Q_{r/2,r^2/2})} \le c\left(\operatorname{vrai\,max}_{Q_{r,r^2}} |u|,\, n,\, \nu,\, \mu\right), \qquad \alpha > 0,$$

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is valid, where

$$|u|_{C_{\alpha,\alpha/2}(Q_{r/2,r^2/2})} = \operatorname{vrai}\max_{(x,t),(x',t')\in Q_{r/2,r^2/2}} \frac{|u(x,t)-u(x',t')|}{|x-x'|^\alpha+|t-t'|^{\alpha/2}},$$

and the constant c also depends on the norms $\|a_i, f_i\|_{L_q(Q_{r,r^2})}$, $\|c, g\|_{L_{q/2}(Q_{r,r^2})}$, $q > n+2$.

3. With the aid of Lemma 1, results analogous to Theorem 3, cases 1), 2), and 3), are obtained for generalized solutions from the class $W_2^1(\Omega)$ of uniformly elliptic equations. Results close to some elliptic analogues of Theorem 3, cases 1) and 2), were obtained in (3^a, 4, 5). A result close to the special case of the theorem corresponding to Theorem 3, case 3), was obtained in (3^b).

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