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Abstract

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ON THE STRUCTURE OF THE COMPLETE CLASS OF UNBIASED ESTIMATORS FOR FAMILIES OF DISTRIBUTIONS OF A SPECIAL FORM

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Let a family of probability measures be given on the space $\{\mathfrak{X}, \mathfrak{A}\}$,

$$P_\theta = \sum_{i=1}^{m} c_i(\theta) P_i, \tag{1}$$

depending on an abstract parameter $\theta \in \Theta$. In all that follows we shall assume that $P_i,\ i = 1, \dots, m$, are mutually absolutely continuous probability measures, and that the functions $c_i(\theta),\ i = 1, \dots, m$, are linearly independent and $c_i(\theta) \geq 0$.

Below we shall be concerned with unbiased estimators ([1]) of functions of the parameter for the family (1). It is obvious that a function $g(\theta)$ can be unbiasedly estimated if and only if

$$g(\theta) = \sum_{i=1}^{m} a_i c_i(\theta), \tag{2}$$

where $a_1, \dots, a_m$ are certain real numbers. In this case every unbiased estimator $\tilde{g}(x)$ of the function $g(\theta)$ satisfies the condition

$$\int_{\mathfrak{X}} \tilde{g}(x)\, dP_i = a_i, \qquad i = 1, \dots, m. \tag{3}$$

The usual measure of quality of an unbiased estimator $\tilde{g}(x)$ of the function $g(\theta)$ is

$$E_\theta[\tilde{g}(x) - g(\theta)]^2 = \int_{\mathfrak{X}} [\tilde{g}(x) - g(\theta)]^2\, dP_\theta. \tag{4}$$

The main result of the present paper consists in a description of the complete class ([2], p. 31) of unbiased estimators of functions of the form (2) for families of distributions (1), when the loss function (4) is used.

Since

$$E_\theta[\tilde{g}(x) - g(\theta)]^2 = \sum_{i=1}^{m} c_i(\theta) \int_{\mathfrak{X}} \tilde{g}(x)^2 \, dP_i - g(\theta)^2$$

and $c_i(\theta) \geq 0$, it is clear that a necessary condition for the admissibility of $\tilde{g}_1(x)$ as an unbiased estimator of $g(\theta)$ is the nonexistence of a function $\tilde{g}(x)$ such that

$$\int_{\mathfrak{X}} \tilde{g}(x) \, dP_i = a_i, \quad i = 1, \dots, m; \qquad \int_{\mathfrak{X}} \tilde{g}(x)^2 \, dP_i \leq \int_{\mathfrak{X}} \tilde{g}_1(x)^2 \, dP_i, \quad i = 1, \dots, m, \tag{5}$$

with strict inequality for at least one i. Fix m numbers $a_1, \dots, a_m$, and denote by L the set of functions $\tilde{g}(x)$ on $\{\mathfrak{X}, \mathfrak{A}\}$ satisfying⋯

satisfying conditions (3) and

$$\int_{\mathfrak{X}} \tilde{g}(x)^2 \, dP_i < \infty, \qquad i = 1, \dots, m. \tag{6}$$

We now proceed to describe, for a nonempty subset $L_1 \subset L$, all such functions $\tilde{g}_1(x)$ that, if $\tilde{g}(x) \in L$ and $\tilde{g}(x) \neq \tilde{g}_1(x)$, then the inequalities

$$\int_{\mathfrak{X}} \tilde{g}_1(x)^2 \, dP_i \geq \int_{\mathfrak{X}} \tilde{g}(x)^2 \, dP_i, \qquad i = 1, \dots, m, \tag{7}$$

cannot hold simultaneously.

Consider the Hilbert space H of all functions square-summable with respect to the measure

$$P = \frac{1}{m} \sum_{i=1}^{m} P_i;$$

it consists precisely of the functions satisfying condition (6). Introduce the operators

$$A_k h(x) = p_k(x) h(x), \tag{8}$$

where $p_k(x) = dP_k/dP, \; h \in H$. Obviously,

$$0 \le p_k(x) \le m, \quad \sum_{k=1}^{m} p_k = m, \quad \|A_k\| \le m, \quad \int_{\mathfrak{X}} h(x)^2 \, dP_k = (A_{kh}, h),$$

$$\int_{\mathfrak{X}} h(x) \, dP_k = (h, p_k), \qquad L = \bigcap_{k=1}^{m} \{h \in H : \ (h, p_k) = a_k\}.$$

Now let $A_1, \dots, A_m$ be some bounded self-adjoint positive operators, and $p_1, \dots, p_m$ some functionals (we ignore the connection, due to the condition, between A_k and p_k). By mutual absolute continuity of the measures P_k, the value $\lambda = 0$ is not an eigenvalue of any A_k. Let

$$S_k(\tilde{g}_1) = L \cap \{h : \ (A_{kh}, h) \le (A_k \tilde{g}_1, \tilde{g}_1)\}. \tag{9}$$

Since $\lambda = 0$ is not an eigenvalue of the operators A_k, the ellipsoids $S_k(\tilde{g}_1)$ are strictly convex. Therefore either their intersection consists of the single element $\tilde{g}_1$, or the intersection of the interiors of these ellipsoids is nonempty, and then the estimator $\tilde{g}_1(x)$ is not an admissible estimator of $g(\theta)$.

We shall indicate all $\tilde{g}_1$ for which

$$\bigcap_{k}^{m} S_k(\tilde{g}_1) = \{\tilde{g}\}.$$

Each of the ellipsoids S_k, unless it degenerates into a point, contains, in view of the boundedness of the operator A_k, some open set and therefore in the affine space L has a supporting hyperplane passing through the point $\tilde{g}_1$. Let T_k be the open half-space of the space L lying on the same side of this hyperplane as the ellipsoid S_k; in the case when S_k degenerates into a point, we agree to set $T_k = L$.

Lemma 1. $\bigcap_{k=1}^{m} S_k = \{\tilde{g}_1\}$ if and only if

$$\bigcap_{k=1}^{m} T_k = \emptyset.$$

Indeed, the cone $\bigcap_{k=1}^{m} T_k$ is the cone generated by the set

$$\bigcap_{k=1}^{m} \operatorname{int} S_k = \operatorname{int} \bigcap_{k=1}^{m} S_k.$$

Each half-space T_k is characterized by an outer normal n_k (emanating from $\tilde{g}_1$).

Lemma 2. $\bigcap_{k=1}^{m} T_k = \emptyset$ if and only if $\Gamma\{l_k\}$ contains a subspace different from a point, where l_k are the rays generated by the vector n_k; Γ is the convex hull.

The formulation of the lemma also covers the degenerate case, if one agrees to regard $l_k = L$ when $T_k = \emptyset$.

Proof. If $\Gamma\{l_k\}$ contains some straight line l, then the intersection of the sets $\overline{T}_k$ is contained in the orthogonal complement to l and therefore contains no open subset; the same property in this case is possessed also by $\bigcap_{k=1}^{m} T_k$, i.e., it is empty. Conversely, if $\bigcap_{k=1}^{m} T_k = \emptyset$, then $\bigcap_{k=1}^{m} \overline{T}_k$ is contained in a proper subspace of the space L. In this case $\Gamma\{l_k\}$ contains its orthogonal complement.

Let us now write the vectors n_k explicitly. The vector n_k is the vector of the (external) normal to the ellipsoid S_k in L. Consider the subspace $L_0 \subset H$, $L_0 = L - \tilde{g}_1$. Let Q_k be the projection operator onto L_0, orthogonal in the sense of the quadratic form (A_{kh}, h), i.e., such that if $(A_{kh}, \tilde{g}_0) = 0$ for all $\tilde{g}_0 \in L_0$, then $Q_{kh} = 0$. Under projection by means of Q_k, the vector $\tilde{g}_1$ passes into $Q_k\tilde{g}_1 \in L_0$, while the vector n_k preserves its direction. Since the vector $Q_k\tilde{g}_1$ is orthogonal to the supporting plane in L_0 to the ellipsoid $Q_{kS}k$ in the sense of the quadratic form $(\Pi_{kA_{kh}}, h)$ (where Π_k is the ordinary orthogonal projection operator onto L_0), and n_k is orthogonal to this supporting plane in the ordinary sense, we obtain

$$n_k = \Pi_{kA_{kQ}} k\tilde{g}_1. \tag{10}$$

Thus, $\tilde{g}_1 \in L_1$ if and only if the convex cone spanned by the vectors $\Pi_{kA_{kQ}} k\tilde{g}_1$ contains some straight line. Here, to compute the operator Q_k, one must solve a finite system of linear equations.

To the description of L_1 one may also add the following. Let

$$r_k(\tilde{g}) = (A_{kQ}k\tilde{g}, Q_k\tilde{g})^{1/2};$$

then $r(\tilde{g}) = \{r_1(\tilde{g}), \dots, r_m(\tilde{g})\}$ maps L continuously into R^m, and it is not difficult to see that each $\tilde{g}_1 \in L_1$ is uniquely determined by its image and even by its $(n-1)$ coordinates. Thus, L_1 is homeomorphic to a bounded subset of R^m of dimension not exceeding $(n-1)$.

In conclusion we make two remarks.

Remark 1. The results set out above can be transferred to the case when the measures P_k are not mutually absolutely continuous.

Remark 2. We intend to discuss separately the application of the results obtained to the analysis of estimates for certain families of distributions of the form (1).

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Note: Figure translations are in progress. See original paper for figures.