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ON THE REDUCTION METHOD FOR SYSTEMS OF EQUATIONS OF WIENER-HOPF TYPE

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The present communication is devoted to extending to the case of systems of equations the principal results of the note (1). In particular, here a justification is given of the reduction method for various systems of integral equations with kernels depending on the difference of the arguments, their discrete analogues, and systems of singular integral equations on the unit circle.

1. Let E (E^+) be one of the spaces l_p $(1 \le p < \infty)$; c_0, the space of complex sequences $\xi = \{\xi_j\}_{-\infty}^{\infty}$ $(\xi = \{\xi_j\}_0^{\infty})$; E_n, the space of sequences of n-dimensional vectors whose coordinates with the same number j $(j = 1, 2, \dots, n)$ form a sequence from E (E^+), with the natural definition of the norm; l_1^n, the set of sequences of matrices of order n whose corresponding elements form a sequence from l_1. To each element $\{A_j\}_{-\infty}^{\infty} \in l_1^n$ there is assigned a matrix-function continuous on the unit circle

$$\mathcal{A}(\zeta) = \sum_{j=-\infty}^{\infty} A_j \zeta^j \quad (|\zeta| = 1). \tag{1}$$

Theorem 1. Let $\{A_j\}_{-\infty}^{\infty} \in l_1^n$. In order that, starting with some m, the system of equations

$$\sum_{k=-m}^{m} A_{j-k} \xi_k = \eta_j \quad (j = 0, \pm 1, \dots, \pm m) \tag{2}$$

have a unique solution $\{\xi_j^{(m)}\}_{j=-m}^{m}$, and that the sequence of vectors $\xi^{(m)} = \{\tilde{\xi}_j^{(m)}\}_{j=-\infty}^{\infty}$, where $\tilde{\xi}_j^{(m)} = \xi_j^{(m)}$ for $|j| \le m$, $\tilde{\xi}_j^{(m)} = 0$ for $|j| > m$, converge as $m \to \infty$ in the norm of the space E_n, whatever the vector $\{\eta_j\}_{-\infty}^{\infty} \in E_n$, it is necessary and sufficient that: a) $\det \mathcal{A}(\zeta) \ne 0$ $(|\zeta| = 1)$; b) the left and right indices of the matrix-function $\mathcal{A}(\zeta)$ be equal to zero[*].

When conditions a) and b) are fulfilled, the sequence of vectors $\xi^{(m)}$ converges to the solution of the system

$$\sum_{k=-\infty}^{\infty} A_{j-k}\xi_k = \eta_j \quad (j = 0, \pm 1, \dots).$$

We note that the fulfillment of conditions a) and b) already follows from the fact that, starting with some m, system (2) is uniquely solvable and the estimate $|\xi^{(m)}| \leq c|\eta^{(m)}|$ holds, where $\eta^{(m)} = \{\eta_j^{(m)}\}_{-\infty}^{\infty}$, $\eta_j^{(m)} = \eta_j$ for $|j| \leq m$, $\eta_j^{(m)} = 0$ for $|j| > m$, and the constant $c(> 0)$ depends only on $\{A_j\}_{-\infty}^{\infty}$.

* For the definition of the left and right indices see (^{2}). We note that, in the terminology of the monograph (^{3}), the left indices of the matrix-function $\mathcal{A}(\xi)$ are its partial indices, and the right ones are the partial indices of the transposed matrix-function.

A theorem analogous to Theorem 1 holds for the discrete analogue of a system of Wiener–Hopf integral equations, i.e., for the infinite system

$$\sum_{k=0}^{\infty} A_{j-k}\xi_k = \eta_j \qquad (j = 0, 1, \dots). \tag{3}$$

We give its formulation.

Theorem 2. *Let* $\{A_j\}_{-\infty}^{\infty} \in l_1^n$. *In order that, beginning with some* m, *the system of equations*

$$\sum_{k=0}^{m} A_{j-k}\xi_k = \eta_j \qquad (j = 0, 1, \dots, m)$$

have a unique solution $\{\xi_j^m\}_{j=0}^m$, *and that the sequence of vectors*

$$\xi_j^{(m)} = \{\xi_0^{(m)}, \xi_1^{(m)}, \dots, \xi_m^{(m)}, 0, 0, \dots\}$$

converge as $m \to \infty$, *in the norm of the space* E_n^+, *whatever the vector* $\{\eta_j\}_0^\infty \in E_n^+$, *it is necessary and sufficient that conditions a) and b) of Theorem 1 be satisfied. If these conditions are satisfied, the sequence of vectors* ξ^m *converges to the solution of system (3).*

The sufficiency of the conditions of Theorem 2 for the case $n = 1$ was established by G. Baxter (^{4}).

In Theorems 1 and 2 the condition $\{A_j\}_{-\infty}^{\infty} \in l_1^n$ can be weakened. For example, in the case $E = l_2$ ($E^+ = l_2^+$), the necessity of conditions a) and b) of these theorems can be proved if only A_j $(j = 0, \pm 1, \dots)$ are such that the matrix-function $\mathcal{A}(\xi)$, defined by equality (1), is continuous (in this case the left and right indices are understood in the sense of paper (^{5}); see also (^{6})). In the proof of the sufficiency of the conditions of Theorems 1 and 2 one uses only the fact that $\mathcal{A}(\xi)$ is a continuous matrix-function admitting a left and a right factorization (see (^{2})).

2. In this section E denotes one of the Banach spaces $L_p(-\infty, \infty)$ $(1 \le p < \infty)$, $C_0(-\infty, \infty)$; E_n is the space of n-dimensional vectors with components from E, and L_1^n is the set of matrices of order n with elements from $L_1(-\infty, \infty)$. The continuous analogue of Theorem 1 is the following.

Theorem 3. *Let the matrix-function $k(t) \in L_1^n$. In order that, beginning with some $\tau(> 0)$, the equation*

$$\varphi(t) - \int_{-\tau}^{\tau} k(t-s)\varphi(s)\,ds = f(t) \qquad (-\tau \le t \le \tau)$$

have a unique solution $\varphi_\tau(t)$, whatever the vector-function $f(t) \in E_n$, and that the vector-functions

$$\widetilde{\varphi}_\tau(t) = \begin{cases} \varphi_\tau(t), & |t| \le \tau, \\ 0, & |t| > \tau, \end{cases}$$

converge as $\tau \to \infty$ in the norm of the space E_n, it is necessary and sufficient that

$$\det(I - K(\lambda)) \ne 0 \qquad (-\infty < \lambda < \infty),$$

where $K(\lambda)$ is the Fourier transform of $k(t)$, and that the left and right indices of the matrix-function $I - K(\lambda)$ be equal to zero. If these conditions are satisfied, the vector-functions $\widetilde{\varphi}_\tau(t)$ converge to the solution of the equation

$$\varphi(t) - \int_{-\infty}^{\infty} k(t-s)\varphi(s)\,ds = f(t) \qquad (-\infty < t < \infty).$$

The continuous analogue of Theorem 2 is formulated in an analogous way. We also note that the remarks to Theorems 1 and 2 remain valid for the continuous analogues of these theorems.

3. In this section we shall adhere to the definitions and notation of §§ 1, 2 of note (^{1}). In addition, by $\mathfrak{B}_n$ we denote the space of all n-dimensional vectors with components in $\mathfrak{B}$, and by $\mathfrak{R}_n(V)$ the normed ring of all matrices of order n with elements from $\mathfrak{R}(V)$. The elements of the ring $\mathfrak{R}_n(V)$ may be regarded as linear bounded operators acting in $\mathfrak{B}_n$. To each operator $\mathcal{A} = \|A_{jk}(V)\|_1^n \in \mathfrak{R}_n(V)$ there corresponds a matrix-function $\mathcal{A}(\xi) = \|A_{jk}(\xi)\|_1^n$, continuous on the unit circle $(|\xi| = 1)$. The ring of

all such matrix-functions with norm $|\mathcal{A}(\xi)| = |\mathcal{A}(V)|$ is denoted by $\mathfrak{R}_n(\xi)$. By $\mathfrak{R}_n^+(\xi)$ ($\mathfrak{R}_n^-(\xi)$) is denoted the closure, in the norm of the ring $\mathfrak{R}_n(\xi)$, of all matrix polynomials of the form

$$\mathcal{A}_0 + \mathcal{A}_1\xi + \ldots + \mathcal{A}_k\xi^k \quad (\mathcal{A}_0 + \mathcal{A}_1\xi^{-1} + \ldots + \mathcal{A}_k\xi^{-k}).$$

As in (^{1}), it is assumed that the number $\dim \mathfrak{B}/V\mathfrak{B}$ is finite, and the projectors $P_\tau \in \Omega(\mathfrak{B})$ converge strongly to the identity operator as $\tau \to \infty$ and satisfy the conditions $P_\tau V P_\tau = P_\tau V$, $P_\tau V^{(-1)} P_\tau = V^{(-1)} P_\tau$ ($\tau \in \Lambda$). Denote by $\mathcal{P}_\tau$ ($\tau \in \Lambda$) the projector defined in the space $\mathfrak{B}_n$ by the matrix $\|P_\tau \delta_{jk}\|_1^n$.

Theorem 4. Let $\mathcal{A} \in \mathfrak{R}_n(V)$, $\det \mathcal{A}(\xi) = 0$ ($|\xi| = 1$), and let $\mathcal{A}(\xi)$ admit left and right factorizations with factors from $\mathfrak{R}_n^\pm(\xi)$. If the left and right indices of the matrix-function $\mathcal{A}(\xi)$ are equal to zero, then, beginning with some τ, the operators $\mathcal{P}_\tau \mathcal{A} \mathcal{P}_\tau$ are invertible in the subspace $\mathcal{P}_\tau \mathfrak{B}_n$, and the operators* $(\mathcal{P}_\tau \mathcal{A} \mathcal{P}_\tau)^{-1}$ converge strongly, as $\tau \to \infty$, to the operator $\mathcal{A}^{-1}$.

4. Let now the operators V, Q_1, Q_2, and P_τ ($\tau \in \Lambda$) satisfy the conditions of § 4 of note (^{1}). Denote by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the projectors defined in $\mathfrak{B}_n$, respectively, by the matrices $\|Q_1 \delta_{jk}\|_1^n$ and $\|Q_2 \delta_{jk}\|_1^n$.

Theorem 5. Let $\mathcal{A}$ and $\mathcal{B}$ be operators from $\mathfrak{R}_n(V)$ satisfying the following conditions: a) $\det \mathcal{A}(\xi) \neq 0$, $\det \mathcal{B}(\xi) \neq 0$ ($|\xi| = 1$); b) the matrix-functions $\mathcal{A}(\xi)$ and $\mathcal{B}(\xi)$ admit left and right factorizations with factors from $\mathfrak{R}_n^\pm(\xi)$, and their left and right indices are equal to zero; c) the matrix-function $\mathcal{B}^{-1}(\xi)\mathcal{A}(\xi)$ admits a right factorization with factors from $\mathfrak{R}_n^\pm(\xi)$, and its right indices are equal to zero. Then, beginning with some τ, the operators

$$\mathcal{P}_\tau(\mathcal{A}\mathcal{Q}_1 + \mathcal{B}\mathcal{Q}_2)\mathcal{P}_\tau$$

are invertible in the subspace $\mathcal{P}_\tau \mathfrak{B}_n$, and the operators

$$[\mathcal{P}_\tau(\mathcal{A}\mathcal{Q}_1 + \mathcal{B}\mathcal{Q}_2)\mathcal{P}_\tau]^{-1}$$

converge strongly to the operator

$$(\mathcal{A}\mathcal{Q}_1 + \mathcal{B}\mathcal{Q}_2)^{-1}$$

as $\tau \to \infty$.

In the proofs of Theorems 4 and 5, an essential role is played by the lemma from note (^{1}).

5. The results of §§ 3 and 4 admit a number of applications to certain concrete classes of equations. From Theorem 4 one derives the sufficiency of the conditions of Theorems 1–3. As consequences of Theorem 5 one can obtain various theorems on the reduction of systems of paired integral equations, transposed systems, their discrete analogues, and systems of singular integral equations on the unit circle.

As an example, we give the formulation of a theorem on the reduction of the system of singular integral equations

$$a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\xi)}{\xi-t}\,d\xi=f(t)\qquad(t\in\Gamma),\tag{4}$$

where Γ is the unit circle; $a(t)$ and $b(t)$ $(t\in\Gamma)$ are continuous matrix-functions of order n; $\varphi(t), f(t)\in E_n$, and E_n is the space of n-dimensional vector-functions with components from $L_p(\Gamma)$ $(1<p<\infty)$. The sum and difference of the matrix-functions $a(t)$ and $b(t)$ are denoted, respectively, by $c(t)$ and $d(t)$, and the "matrix" Fourier coefficients of the matrix-functions $c(t)$ and $d(t)$ by c_j $(j=0,\pm1,\dots)$ and d_j $(j=0,\pm1,\dots)$.

* By $(\mathcal{P}_\tau\mathcal{A}\mathcal{P}_\tau)^{-1}$ is denoted the operator equal to the inverse of the operator $\mathcal{P}_\tau\mathcal{A}\mathcal{P}_\tau$ on the subspace $\mathcal{P}_\tau\mathfrak{B}_n$ and equal to zero on the subspace $(I-\mathcal{P}_\tau)\mathfrak{B}_n$. We also note that, under the conditions stated in the theorem, the operator $\mathcal{A}$ is invertible (see (7)).

Theorem 6. Suppose the following conditions are satisfied: a) $\det c(t)\neq 0$, $\det d(t)\neq 0$ $(|t|=1)$; b) the matrix functions $c(t)$ and $d(t)$ admit left and right factorizations, and their left and right indices are equal to zero; c) the matrix function $d^{-1}(t)c(t)$ admits a right factorization and its right indices are equal to zero. Then, beginning with some m, the system of equations

$$\sum_{k=0}^{m}c_{j-k}\varphi_k+\sum_{k=-m}^{-1}d_{j-k}\varphi_k=f_j\qquad(j=0,\pm1,\dots,\pm m),$$

where f_j are the Fourier coefficients of the vector function $f(t)$, has a unique solution $\{\varphi_k^{(m)}\}_{k=-m}^{m}$, and the sequence of vector functions

$$\varphi^{(m)}(t)=\sum_{k=-m}^{m}\varphi_k^{(m)}t^k$$

converges as $m\to\infty$, in the norm of the space E_n, to the solution of equation (4).

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