A METHOD FOR CALCULATING AN INHOMOGENEOUS DIAPHRAGMED RADIO WAVEGUIDE OF FINITE LENGTH

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Abstract

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A METHOD FOR CALCULATING AN INHOMOGENEOUS DIAPHRAGMED RADIO WAVEGUIDE OF FINITE LENGTH

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Any inhomogeneous cellular waveguide may be regarded as a chain of nonidentical functional cells $j = 1, 2, \dots, R$, transforming the distribution functions of the amplitudes of forced oscillations of the tangential fields $\mathbf{E}^{(1)}_{\tau_j}(q)$ and $\mathbf{H}^{(1)}_{\tau_j}(q)$, $q \in S^{(1)}(j)$, at the input apertures $S^{(1)}(j)$ of the cells into the functions $\mathbf{E}^{(2)}_{\tau_j}(q)$ and $\mathbf{H}^{(2)}_{\tau_j}(q)$, $q \in S^{(2)}(j)$, at the output apertures $S^{(2)}(j)$ ([1]). To construct the N-th matrix approximation, let us expand these fields in the basis functions $\{e^{(1,2)}_{k,j}(q)\}^{\infty}_{k=1}$ and $\{h^{(1,2)}_{k,j}(q)\}^{\infty}_{k=1}$, $q \in S^{(1,2)}(j)$, and, taking the first N Fourier coefficients of these expansions $\{V_k(j)\}^N_{k=1}$ and $\{I_k(j)\}^N_{k=1}$ as the amplitudes specified at the strips $k = 1, 2, \dots, N$ of the multiport with number j, we obtain, in place of the waveguide, a chain of $4N$-ports. The forced oscillations in it, in source representation, will be:

$$\left\|\begin{matrix}\mathbf{V}(j)\\ \mathbf{I}(j)\end{matrix}\right\| e^{i\omega t} = e^{i\omega t}\left\{\sum_{j'=0}^{j-1} G'(j,j')Z_1(j') + \sum_{j'=j}^{R-1} G''(j,j')Z_2(j')\right\}\left\|\begin{matrix}\mathbf{V}^0(j')\\ \mathbf{I}^0(j')\end{matrix}\right\|, \quad (1)$$

where $\mathbf{V}(j)$ and $\mathbf{I}(j)$ are N-dimensional vectors with components $\{V_k\}^N_1$ and $\{I_k\}^N_1$. If the ports of adjacent cells are matched, i.e. $e^{(1)}_{k,j+1} = e^{(2)}_{k,j}$, $h^{(1)}_{k,j+1} = h^{(2)}_{k,j}$, then

$$\begin{aligned} G'(j,j') &= A(j-1)\dots A(j'+1)\,\mathcal{G}(j'); \\ G''(j,j') &= A^{-1}(j)\dots A^{-1}(j'-1)A^{-1}(j'), \end{aligned} \quad (2)$$

where $A(j)$ is the $2N \times 2N$ Brillouin matrix of the j-th cell. The Green matrix operator (2) acts on the vectors of external voltages $\mathbf{V}^0(j')$ and currents $\mathbf{I}^0(j')$ through the matrices Z_1 and Z_2:

Fig. 1

Figure 1: Fig. 1

$$Z_1(j') = \begin{pmatrix} \widetilde{Z}(R+1, j'+1)\,\Delta Z^{-1} & -\widetilde{Z}(R+1, j'+1)\,\Delta Z^{-1}\widetilde{Z}(0, j'+1) \\ \Delta Z^{-1} & -\Delta Z^{-1}\widetilde{Z}(0, j'+1) \end{pmatrix}, \tag{3}$$

where $Z(0, j'+1)$ and $Z(R+1, j'+1)$ are the input-impedance matrices $Z(0)$ and $Z(R+1)$ of the end cells $j = 0$ and $j = R+1$, recalculated from the ends of the chain to the section $S^{(1)}(j'+1)$, and $\Delta Z = \widetilde{Z}(0, j'+1) + \widetilde{Z}(R+1, j'+1)$. $Z_2(j')$ is obtained from (3) by interchanging $\widetilde{Z}(0, j'+1)$ and $\widetilde{Z}(R+1, j'+1)$. The introduction of Z_1 and Z_2 removes the pathological properties of the Brillouin matrix as $N \to \infty$ ([1]), and also ensures in (1) the automatic fulfillment of the boundary conditions for $j = 0$ and $j = R+1$. Equation (1) is valid for arbitrary cells with matrices $A(j)$ possessing inverses $A^{-1}(j)$, and

for any frequencies ω, except for the resonant ω_s, determined from the equation $|\Delta Z| = 0$, whose roots do not depend on j'. For arbitrary cells, (1) is an expansion in generalized normal waves of rank $2N$. The appearance in (1) of waves of lower rank is possible for cells possessing identity elements (2). When all $A(j)$ are identical (identical cells) and Z_1 and Z_2 commute, then from (1) we obtain an expansion in ordinary normal waves of rank 1.

Fig. 1. An inhomogeneous cellular waveguide and its equivalent multiconductor network. Unshaded boxes ($j = 0, 1, 2, 3, \dots, R+1$) are the principal cells with Breizig matrices $A(j)$ (formula (9)); for $j = 1, 2, \dots, R$, these are terminal cells, and $j = 0$ and $j = R+1$ are cells with impedance matrices $Z(0)$ and $Z(R+1)$; shaded boxes are matching cells at the junctions of the principal cells ($j = 1, 2, \dots, R+1$) with matrices $A_{\text{match}}(j)$ (formula (11)).

Construct (1) for axially symmetric forced oscillations of the TH type in an inhomogeneous iris-loaded waveguide of circular cross section (Fig. 1). In the case when all $b_j = b$, it can be represented as a chain of cells of two kinds: symmetric cells with diaphragms, characterized by the parameters $a_j, t_j, d_j = D_j - t_j$, and cells without diaphragms ($a_j = b,\ t = 0$). The latter are sections of a cylindrical tube of radius b and length $D_j = d_j$. Taking for them as basis functions $\{e_k\}_k$ and $\{h_k\}_k$ the forms of the normal waves TH_{0k}, i.e.

$$e_k(r/b) = J_1[\rho_{0k}(r/b)]/b\sqrt{\pi}J_1(\rho_{0k}),$$

where J_1 is the Bessel function of order 1, and ρ_{0k} is the k-th root of $J_0(\rho)$, we write in this basis the Breizig matrices in the form

$$A(j) = \begin{pmatrix} \left\|\operatorname{ch}\gamma_k d_j \delta_{kk'}\right\|^{NN} & \left\|-(\gamma_k/i\omega\varepsilon)\operatorname{sh}\gamma_k d_j \delta_{kk'}\right\|^{NN} \\ \left\|-(i\omega\varepsilon/\gamma_k)\operatorname{sh}\gamma_k d_j \delta_{kk'}\right\|^{NN} & \left\|\operatorname{ch}\gamma_k d_j \delta_{kk'}\right\|^{NN} \end{pmatrix}, \tag{4}$$

where

$$\gamma_k = \sqrt{(\rho_{0k}/b)^2 - \omega^2/c^2}$$

is the wave number of wave number k; $\delta_{kk'}$ is the Kronecker symbol. For cells with diaphragms we introduce the basis system of functions $\mathcal{E}_s = \mathcal{E}'_s(r/a)$, $r < a$; $\mathcal{E}_s = 0$, $a < r < b$, $s = 1, 2, \ldots, N$, in sections $\langle 2\rangle$ and $\langle 4\rangle$ (Fig. 1), where the sharp edges of the diaphragms are located, and take into account in $\mathcal{E}'_s$ for $r = a$ the electrostatic singularities of the field E_r.

Expand the fields E_r in sections $\langle 2\rangle$ and $\langle 4\rangle$ in the functions $\mathcal{E}_s$:

$$E_r \simeq \sum_{s=1}^{N} K_s^{(1,2)} \mathcal{E}_s(r).$$

The vectors $\mathbf{K}_1$ and $\mathbf{K}_2$ with components $\{K_s^{(1)}\}^N$ and $\{K_s^{(2)}\}^N$, according to the first boundary-value problem of electrodynamics, uniquely determine the fields of the cell. However, the junction of cells in the chain occurs in the aperture sections $S^{(1)}$ and $S^{(2)}$ ($\langle 1\rangle$ and $\langle 5\rangle$ in Fig. 1); therefore the basis system $\{\mathcal{E}_s\}_1^N$ in $\langle 2\rangle$ and $\langle 4\rangle$ must be transformed to sections $\langle 1\rangle$ and $\langle 5\rangle$. As a result of solving the first problem of electrodynamics (1), we obtain the following relations of the vectors $\mathbf{K}^{(1,2)}$

with the vectors $\mathbf{V}$ and $\mathbf{I}$ in sections $\langle 1\rangle$ (output) and $\langle 5\rangle$ (output)

$$\begin{pmatrix} v_{11} & -v_{12} \\ v_{12} & -v_{11} \end{pmatrix} \left\| \begin{matrix} \{K_s^{(1)}\}_1^N \\ \{K_s^{(2)}\}_1^N \end{matrix} \right\| = \left\| \begin{matrix} \{V_s^{\text{in}}\}_1^N \\ \{V_s^{\text{out}}\}_1^N \end{matrix} \right\|; \qquad \begin{pmatrix} i_{11} & -i_{12} \\ i_{12} & -i_{11} \end{pmatrix} \left\| \begin{matrix} \{K_s^{(1)}\}_1^N \\ \{K_s^{(2)}\}_1^N \end{matrix} \right\| = \left\| \begin{matrix} \{I_s^{\text{in}}\}_1^N \\ \{I_s^{\text{out}}\}_1^N \end{matrix} \right\|. \tag{5}$$

Here v_{11} and v_{12} are matrices with elements

$$v_{ss'}^{11} = \sum_{k=1}^{\infty} \frac{\alpha_{ks}\alpha_{ks'} \ \operatorname{cth} \alpha_k t}{\alpha_k} + \sum_{k=1}^{\infty} \frac{\beta_{ks}\beta_{ks'} \ \operatorname{cth} \gamma_k d/2}{\gamma_k}; \qquad \alpha_k = \sqrt{\left(\frac{\rho_{0k}}{a}\right)^2 - \frac{\omega^2}{c^2}};$$

$$v_{ss'}^{12} = \sum_{k=1}^{\infty} \frac{\alpha_{ks}\alpha_{ks'}}{\alpha_k \ \operatorname{sh} \alpha_k t}; \qquad \alpha_{ks} = \left(\mathcal{E}_s^3,\, e_k\left(\frac{r}{a}\right)\right); \qquad \beta_{ks} = \left(\mathcal{E}_s,\, e_k\left(\frac{r}{b}\right)\right),$$

and the matrices i_{11} and i_{12} have elements

$$i_{ss'}^{11} = v_{ss'}^{11} - \sum_{k=1}^{\infty} \frac{\beta_{ks}\beta_{ks'}}{\gamma_k \ \operatorname{ch} \gamma_k \frac{d}{2} \ \operatorname{sh} \gamma_k \frac{d}{2}}; \qquad i_{ss'}^{12} = v_{ss'}^{12}.$$

The index j is omitted. The coefficients α_{ks} and β_{ks} for $\mathcal{E}'_s = (r/a)^{2s-1}/\sqrt{1-(r/a)^2}$ are given in (3).

The transformation of the components $\{V_s\}_1^N$ and $\{I_s\}_1^N$ of the vectors **V** and **I**, specified in the basis $\{\mathcal{E}_s\}_1^N$ recalculated into $\langle 1\rangle$ or $\langle 5\rangle$, into components of these same vectors specified in the natural basis $\{e_k(r/b)\}$, i.e., into the components $V_k = (E_r, e_k(r/b))$ and $I_k = (H_\varphi, e_k(r/b))$, is expressed by the $N \times \infty$ matrices $\|\beta_{ks}^V\|$ and $\|\beta_{ks}^I\|$:

$$\{V_s\}_1^N = \|\beta_{ks}^V\|^{N,\infty}\{V_k\}_1^\infty; \qquad \{I_s\}_1^N = \|\beta_{ks}^I\|^{N,\infty}\{I_k\}_1^\infty, \tag{6}$$

$$\beta_{ks}^V = \beta_{ks}/\gamma_k \operatorname{sh}\gamma_k d/2 \quad \text{and} \quad \beta_{ks}^I = \beta_{ks}/i\omega\varepsilon\gamma_k \operatorname{ch}\gamma_k d/2.$$

From (5) we obtain the admittance matrix of a cell with a diaphragm in the basis of functions $\{\mathcal{E}_s\}_1^N$, recalculated to the output and input sections of the cell:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} i_{11} & -i_{12} \\ i_{12} & -i_{11} \end{pmatrix} \begin{pmatrix} v_{11} & -v_{12} \\ v_{12} & -v_{11} \end{pmatrix}^{-1}. \tag{7}$$

Here

$$Y_{11} = i_{11}\Delta - v_{12}\Delta v_{12}v_{11}^{-1}; \qquad Y_{12} = i_{11}\Delta v_{12}v_{11}^{-1} - v_{12}\Delta, \tag{8}$$

$$\Delta = (v_{11} - v_{12}v_{11}^{-1}v_{12})^{-1};$$

$Y_{22} = Y_{11}$ by symmetry of the cell, and $Y_{21} = Y_{12}$ owing to the satisfaction of the reciprocity principle. The Bracewell matrix of the cell in this same basis has the form

$$A(j) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} -Y_{12}^{-1}Y_{11} & Y_{12}^{-1} \\ Y_{11}Y_{12}^{-1}Y_{11} - Y_{12} & -Y_{11}Y_{12}^{-1} \end{pmatrix}, \tag{9}$$

where $A_{11}A_{21}^{-1}A_{22}A_{21} - A_{12}A_{21} = \mathcal{E}$.

The input-impedance matrices of the terminal cells are determined by the formulas $Z(0) = Y_{11}^{-1}(0)$; $Z(R+1) = Y_{11}^{-1}(R+1)$, if the input aperture of the 0-th cell and the output aperture of the $(R+1)$-th cell are closed by perfectly conducting partitions. If the cells are nonidentical, then the matrices (9) cannot be used directly to construct the Green' s function (2), since they are written in different bases. Let us reduce them all to the basis $\{e_n(r/b)\}_k$, assuming that the field forms in the apertures of the cells are approximated with sufficient accuracy by N basis functions $\{e_k\}_1^N$. Then, truncating the matrices in (6) to the N-th column and inverting these relations, we obtain

$$\{V_k\}_1^N \simeq \|\bar\beta^V_{ks}\|^{-1}\{V_s\}_1^N; \qquad \{I_k\}_1^N \simeq \|\bar\beta^I_{ks}\|^{-1}\{I_s\}_1^N. \tag{10}$$

Here it is assumed that the matrices $\|\beta^V_{ks}\|$ and $\|\bar\beta^I_{ks}\|$ are nonsingular. Using the abbreviated forms (5) and (10), we obtain that the matching $4N$-terminal network, which must be inserted between the j-th and $(j+1)$-st cells with diaphragms, has the quasi-diagonal Breisig matrix A_{match}

$$\begin{pmatrix} \|\bar\beta^V_{ks}(j+1)\|\cdot\|\bar\beta^J_{ks}(j)\|^{-1} & 0 \\ 0 & \|\bar\beta^V_{ks}(j+1)\|\cdot\|\bar\beta^J_{ks}(j)\|^{-1} \end{pmatrix}. \tag{11}$$

If adjacent cells turn out to be a cell with a diaphragm and a cell without one, then between them one should insert a matching $4N$-terminal network with a quasi-diagonal matrix A_{match} with blocks $\|\beta^V_{ks}(j)\|$ and $\|\beta^J_{ks}(j)\|$, or with blocks $\|\bar\beta^V_{ks}(j)\|^{-1}$ and $\|\beta^J_{ks}(j)\|^{-1}$, depending on the order in which the cells are arranged. The general circuit diagram with matching cells is given in the lower half of Fig. 1. In constructing the Green operator (2), the matrices (4), (9), (11) should be successively multiplied in the order of their arrangement in the diagram of Fig. 1. We note that, in order to obtain the operator expression (1), it is also necessary to compute Z_1 and Z_2 by formula (3). The transformed impedances entering into it are determined by the propagation method with the aid of the formula

$$Z(j_0, j'+1) = [A_{11}Z(j_0) - A_{12}][-A_{21}Z(j_0) + A_{22}]^{-1},$$

where A_{ik} are the blocks of the matrix $G'(j'+1,1)$ for $j_0 = 0$, and of the matrix $G''(j'+1,R)$ for $j_0 = R+1$ (see in more detail ([1])). The vectors $V^0(j')$ and $I^0(j')$ in (1) are determined by expanding the fields $E_r(r)$ and $H_\varphi(r)$, specified at the input cross sections of the cells of the inhomogeneous waveguide, in the corresponding systems of basis functions of the input cross sections.

If the cellular waveguide is infinite and consists of identical cells $A_j = A$, then the matrices $G'(j,j')$ and $G''(j,j'')$ in (1) can be simultaneously diagonalized by the same linear change of basis. In this case there are no matching cells, and the basic cells form a countable set of mutually unconnected chains of two-terminal networks $l = 1,2,3,...$. In each such chain there exists one normal wave with wave number ψ_l, $l = 1,2,...$, determined through the eigenvalue $\lambda_l = \exp(-\psi_l)$ of the matrix A. By virtue of the symmetry of the cells and the fulfillment of the reciprocity principle, the eigenvalue equation for λ_l reduces to the form

$$\det(Y_{11} - \cos\psi\, Y_{12}) = 0. \tag{12}$$

It is one of the modifications of the dispersion equation for a homogeneous cellular waveguide considered in ([3]) (formula (16)).

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Note: Figure translations are in progress. See original paper for figures.