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Abstract

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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE BOLTZMANN EQUATION

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The proof of existence and uniqueness of the solution of problems for the Boltzmann equation has been the subject of works by Carleman, Wild, Morgenstern, and Grad. Carleman ([1]) proved an existence and uniqueness theorem "in the large" (an arbitrary time interval) for the Cauchy problem for an unbounded homogeneous gas consisting of absolutely elastic spheres. An analogous theorem was proved by Wild ([2]) and Morgenstern ([3]) (an improved version of the proof) for a gas of "pseudo-Maxwellian" particles, when the collision integral does not depend on the magnitude of the relative velocity. Grad ([4]) proved an existence and uniqueness theorem "in the small" for the Cauchy problem for an inhomogeneous gas of "pseudo-Maxwellian" particles. In this case the solution is obtained in the class of functions bounded in modulus.

In the present work an existence and uniqueness theorem is proved for the Cauchy problem for the Boltzmann equation describing the flow past a convex body by a rarefied gas of structureless particles under sufficiently general boundary conditions and a broad class of interaction potentials between the gas particles.

The state of a rarefied gas of structureless particles is described by a function $f(\mathbf{r}, \mathbf{u}, t)$, having the meaning of the density of the mathematical expectation of the number of particles in an element of phase-space volume at time t and satisfying the integro-differential Boltzmann equation. In the nonstationary problem of the flow of a gas past a convex body, the integral form of the equation can be written as

$$f(\mathbf{r}, \mathbf{u}, t) = F_0(\mathbf{r}, \mathbf{u}, t) + \int_0^t I(\mathbf{r}, \mathbf{u}, \tau, t)\, d\tau, \tag{1}$$

where

$$F_0(\mathbf{r},\mathbf{u},t) = \begin{cases} F_0'(\mathbf{r},\mathbf{u},t) = f_0(\mathbf{r}-\mathbf{u}t,\mathbf{u}), & \tau_s \leqslant 0 \text{ or does not exist,} \\ \dfrac{1}{|u_n|}\displaystyle\int_{(u_{1n}<0)} |u_{1n}|\, F_0'(\mathbf{r}_s,\mathbf{u}_1(\tau_s))\,\widetilde{T}(\mathbf{u}_1,\mathbf{u})\,d\mathbf{u}_1, & \tau_s > 0; \end{cases}$$

$$I(\mathbf{r},\mathbf{u},\tau,t) = \begin{cases} I'(\mathbf{r},\mathbf{u},\tau,t) = \Big\{\displaystyle\int_{(\mathbf{u}_1)}\int_{(\mathbf{u}_2)} \sigma(\mathbf{u}_1-\mathbf{u}_2)\,|\mathbf{u}_1-\mathbf{u}_2| \\ \quad\times f(\mathbf{r}-\mathbf{u}(t-\tau),\mathbf{u}_1,\tau)\, f(\mathbf{r}-\mathbf{u}(t-\tau),\mathbf{u}_2,\tau) \\ \quad\times T(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u})\,d\mathbf{u}_1\,d\mathbf{u}_2 - f_0(\mathbf{r}-\mathbf{u}(t-\tau),\mathbf{u}) \\ \quad\times \displaystyle\int_{(\mathbf{u}_1)} \sigma(\mathbf{u}-\mathbf{u}_1)\,|\mathbf{u}-\mathbf{u}_1| \\ \quad\times f(\mathbf{r}-\mathbf{u}(t-\tau),\mathbf{u}_1,\tau)\,d\mathbf{u}_1\ \Big\} \\ \quad\times \exp\Big\{-\displaystyle\int_\tau^t\int_{(\mathbf{u}_1)} \sigma(\mathbf{u}-\mathbf{u}_1)\,|\mathbf{u}-\mathbf{u}_1| \\ \quad\times f(\mathbf{r}-\mathbf{u}(t-q),\mathbf{u}_1,q)\,d\mathbf{u}_1\,dq\Big\}, \quad \tau \geqslant \tau_s, \\ \dfrac{1}{|u_n|}\displaystyle\int_{(u_n'<0)} |u_n'|\, I'(\mathbf{r}_s,\mathbf{u}',\tau,\tau_s)\,\widetilde{T}(\mathbf{u}',\mathbf{u})\,d\mathbf{u}', \quad \tau < \tau_s. \end{cases}$$

u_n is the projection of the velocity onto the normal to the surface of the body at the point under consideration; σ is the collision cross-section; T and $\widetilde{T}$ are probabilistic characteristics of the results of collisions of particles with one another and with the boundary; f_0 is the initial distribution function; $F(\mathbf{r}) = 0$ is the equation of the surface of the body; τ_s is the largest root of the equation $F(\mathbf{r}-\mathbf{u}(t-\tau_s)) = 0$, $\mathbf{r}_s = \mathbf{r}-\mathbf{u}(t-\tau_s)$.

The form of the equation presented above is obtained from those known earlier [4-6] by factoring out the initial distribution function f_0 without the exponential and by representing the function f on the boundary for reflected particles in terms of f_0 in accordance with the equation

$$f(\mathbf{r}_s,\mathbf{u},t_s)\big|_{u_n>0} = \frac{1}{|u_n|}\int_{(u_{1n}<0)} |u_{1n}| \left\{ f_0(\mathbf{r}_s-\mathbf{u}_1\tau_s,\mathbf{u}_1) + \int_0^{\tau_s} I'(\mathbf{r}_s,\mathbf{u}_1,\tau,\tau_s)\,d\tau \right\} \widetilde{T}(\mathbf{u}_2,\mathbf{u})\,d\mathbf{u}_1.$$

This form turns out to be more convenient not only in considering questions of existence and uniqueness, but also in the constructive solution of certain nonstationary flow problems.

Let us consider spaces $\mathcal{A}$ of functions $f(\mathbf{r},\mathbf{u},t)$ such that

$$0 \le f(\mathbf{r},\mathbf{u},t) \le \mathbf{f}(\mathbf{u}), \qquad f(\mathbf{r},\mathbf{u},0) = f_0(\mathbf{r},\mathbf{u}),$$

$$0 \le f_0(\mathbf{r},\mathbf{u}) \le \alpha \mathbf{f}(\mathbf{u}), \qquad 0 < \alpha < 1,$$

where $\mathbf{f}$ is the global Maxwellian distribution.

Introduce

$$\|\varphi(\mathbf{r},\mathbf{u},t)\| = \max_{(\mathbf{r},\mathbf{u},t)} \frac{|\varphi(\mathbf{r},\mathbf{u},t)|}{\mathbf{f}(\mathbf{u})}, \qquad \rho(f_1,f_2) = \|f_1 - f_2\|.$$

Let

$$\mu(\mathbf{u}) = \int_{(\mathbf{u}_1)} \sigma(\mathbf{u}-\mathbf{u}_1)|\mathbf{u}-\mathbf{u}_1|\,\mathbf{f}(\mathbf{u}_1)\,d\mathbf{u}_1,$$

$$\lambda(\mathbf{u}) = \frac{1}{|u_n|} \int_{(u_{1n}<0)} |u_{1n}|\,\mathbf{f}(\mathbf{u}_1)\,\widetilde{T}(\mathbf{u}_1,\mathbf{u})\,d\mathbf{u}_1.$$

Theorem. If $\mu(\mathbf{u}) \le A = \text{const} < \infty$, $\lambda(\mathbf{u}) \le B\mathbf{f}(\mathbf{u})$ $(B = \text{const} < \infty)$, then on some interval $t \in [0, t^*]$ there exists a solution of equation (1), unique in the class $\mathcal{A}$. Here $t^* = \min\{t_1, t_2\}$, where t_1 is the positive root of the equation

$$\frac{1}{2}(1+\alpha)A(1+B)t^2 + (2+\alpha)(1+B)t - \varepsilon/A = 0,$$

$$t_2 < [1 - \max\{\alpha, \alpha B\}]/AB(1+\alpha)$$

$(\varepsilon < 1,\ \alpha B < 1)$.

The idea of the proof is based on the preservation of the properties of the initial distribution uniformly in $\mathbf{r}$ and $\mathbf{u}$ over a certain time interval, which is manifested in the dependence of t^* on α, A, B.

Checking the conditions of the contraction mapping principle constitutes the essence of the proof method. For the operator W appearing on the right-hand side of equation (1), the estimates obtained are

$$\|Wf_1 - Wf_2\| \le A\left[\frac{(1+\alpha)}{2}A(1+B)t^2 + (2+\alpha)(1+B)t\right]\|f_1 - f_2\|,$$

$$0 \le Wf \le [\max\{\alpha, \alpha B\} + tAB(1+\alpha)]\mathbf{f},$$

from which the validity of the formulated theorem follows.

The inequality $\mu \leq A$ is valid for any bounded interaction potentials of particles with one another or for potentials tending to infinity no faster than $1/\rho^4$ (ρ is the distance between particles). Potentials more singular at zero lead to singularities in the behavior of $\mu(\mathbf{u})$ at infinitely large velocities. Because the models singular at zero (for example, the Lennard-Jones potential, hard spheres)

balls) describe real interactions sufficiently well only at moderate relative collision energies; the inequality given should be regarded as a purely mathematical restriction. We note, however, that the notion of immutable neutral particles and, consequently, the Boltzmann equation are valid only at moderate relative collision energies. Therefore, the dependence of mathematical results on the character of the behavior of the potentials at zero is, generally speaking, undesirable.

The second inequality, $\lambda \leq Bf(\mathbf{u})$, is equivalent to the uniform, in $\mathbf{u}$, convergence of the integral

$$\int_{(u_{1n}<0)} \frac{|u_{1n}|f(\mathbf{u}_1)}{|u_n|f(\mathbf{u})} \widetilde{T}(\mathbf{u}_1, \mathbf{u})\, d\mathbf{u}_1,$$

which, apparently, is satisfied for all sufficiently realistic models of interaction. In particular, this requirement is met by the widely used scheme of diffuse-specular reflection. From the theorem proved, as a special case, there follows the theorem of existence and uniqueness for the Cauchy problem without boundary.

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Note: Figure translations are in progress. See original paper for figures.