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**Abstract**

**Full Text**

**S. A. Vinogradov**

**ON INTERPOLATION AND ZEROS OF POWER SERIES WITH A SEQUENCE OF COEFFICIENTS FROM  $l^p$**

*(Presented by Academician V. I. Smirnov, 3 VII 1964)*

I. Let  $U$  be the open,  $\bar{U}$  the closed unit disk of the complex plane,  $T \subset \bar{U}$ ; let  $l_A^p$  be the space of all functions  $f$ , regular in  $U$  and such that the norm

$$\|f\|_p = \left( \sum_{k=0}^{\infty} \left| \frac{f^{(k)}(0)}{k!} \right|^p \right)^{1/p} < +\infty \quad (p \geq 1).$$

Denote by the symbol  $I$  the embedding operator from the space  $l_A^1$  into the space of all finite complex functions defined on  $T$ :  $If(z) = f(z)$  for all  $f \in l_A^1$ ,  $z \in T$ . The aim of the first part of this note is, for some sets  $T \subset U$ , to describe the set  $I(l_A^1)$ . An analogous problem for other classes of analytic functions was solved in papers (1-4).

Obviously, all elements of  $I(l_A^1)$  are functions continuous on  $T$ . Following (7), we shall call a set  $T \subset \bar{U}$  a **Carleson set** if  $I(l_A^1) = C_T$ , where  $C_T$  is the space of all functions continuous on  $T$ . In (7) the existence of infinite and even perfect Carleson sets lying on the unit circle was proved.

**Theorem 1.** *Every Carleson set contained in  $U$  is finite.*

Let us introduce some notation:

1°.  $V_A$  is the space of all functions  $f$ , analytic in  $U$  and such that

$$\sup_{0 < r < 1} \int_0^{2\pi} |f'(re^{it})| dt < +\infty.$$

In  $V_A$  the norm

$$\|f\|_{V_A} = \frac{1}{2\pi} |f(1)| + \frac{1}{2\pi} \int_{|\zeta|=1} |f'(\zeta)| |d\zeta|,$$

is introduced, where  $f'(\zeta)$  is the boundary value of the function  $f'$ .

Let us note that  $V_A \subset l_A^1$  (see (6)).

2°.

$$T = \{z_k\}_{k=1}^{\infty}, \quad |z_k| < 1 \quad (k = 1, 2, \dots), \quad z_k \neq z_m, \text{ if } k \neq m.$$

The sequence  $\{z_k\}_{k=1}^{\infty}$  is numbered so that

$$|z_1| \leq |z_2| \leq \dots \leq |z_k| \leq \dots.$$

3°.  $bv_T$  is the space of all functions  $\psi$ , defined on  $T$  and such that the norm

$$\|\psi\|_{bv_T} = |\alpha_0| + \sum_{k=1}^{\infty} |\psi(z_{k+1}) - \psi(z_k)| < +\infty,$$

where

$$\alpha_0 = \lim_{k \rightarrow \infty} \psi(z_k).$$

4°.

$$B_n(z) = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z}, \quad b_n(z) = \prod_{k=n+1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k}z},$$

where  $|z_k| < 1$  ( $k = 1, 2, \dots$ ) and

$$\sum_{k=1}^{\infty} (1 - |z_k|) < +\infty.$$

In the case when

$$T = \{z_k\}_{k=1}^{\infty} \subset [0, 1], \quad z_k < z_{k+1} \quad (k = 1, 2, \dots),$$

we have

$$I(l_A^1) \subset bv_T.$$

Indeed, let  $f \in l_A^1$ ,  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ; then

$$\begin{aligned} \sum_{n=1}^{\infty} |f(z_n) - f(z_{n+1})| &\leq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} |a_k| |z_{n+1}^k - z_n^k| \leq \\ &\leq \sum_{k=0}^{\infty} |a_k| \sum_{n=1}^{\infty} (z_{n+1}^k - z_n^k) \leq \sum_{k=0}^{\infty} |a_k| < +\infty. \end{aligned}$$

Let the set  $T = \{z_k\}_{k=1}^{\infty} \subset U$  satisfy the following condition:  $z_k \rightarrow 1$  as  $k \rightarrow \infty$  along a nontangential path, i.e., there exists  $\lambda \geq 1$  such that

$$|1 - z_k| \leq \lambda(1 - |z_k|) \quad (k = 1, 2, \dots). \quad (*)$$

**Theorem 2.** Let the set  $T = \{z_k\}_{k=1}^{\infty} \subset U$  satisfy conditions 2<sup>0</sup>, (\*). Then the following assertions are equivalent:

1.  $I(l_A^1) = bv_T$ .
2.  $I(V_A) = bv_T$ .

3.

$$\inf_n \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| > 0. \quad (**)$$

Let us note that condition  $(**)$  is a criterion for the solvability of a number of other interpolation problems (see  $(1, 2, 3)$ ).

The proof of Theorems 1 and 2 is based on the following fundamental lemmas.

**Lemma 1.** Let  $T = \{z_k\}_{k=1}^{\infty}$  satisfy conditions  $2^0$ ,  $(*)$ ,  $(**)$ . Then the sequence of functions

$$\Phi_n(z) = \frac{(1 - z_n^2)(1 - z^2)}{(1 - z\bar{z}_n)^2} \frac{B_n(z)}{B_n(z_n)} \quad (n = 1, 2, \dots)$$

satisfies the conditions:

1.  $\{\Phi_n\}_{n=1}^{\infty} \subset V_A$ .
2.  $\Phi_n(z_k) = \delta_{nk}$  ( $\delta_{nk}$  is the Kronecker symbol,  $n, k = 1, 2, \dots$ ).
3.  $\|\Phi_n\|_V < M_1 < +\infty$  ( $n = 1, 2, \dots$ ),  $M_1$  independent of  $n$ .

**Lemma 2.** Under the conditions of Lemma 1, the sequence of functions

$$F_n(z) = \frac{1 - z}{1 - z\bar{z}_n} b_n(z) + \sum_{k=1}^n \left( 1 - \frac{1 - z_k}{1 - z_k \bar{z}_n} b_n(z_k) \right) \Phi_k(z) \quad (n = 1, 2, \dots),$$

where  $\Phi_k(z)$  is from Lemma 1, satisfies the conditions:

1.  $\{F_n\}_{n=1}^{\infty} \subset V_A$ .
2.  $F_n(z_k) = 1$ , if  $k \leq n$ ;  $F_n(z_k) = 0$ , if  $k > n$ .
3.  $\|F_n\|_V < M_2 < +\infty$  ( $n = 1, 2, \dots$ ),  $M_2$  independent of  $n$ .

**Corollary.** Under the conditions of Lemma 1 we have: to every function  $\psi \in bv_T$  there corresponds a function

$$f(z) = \alpha_0 + \sum_{k=1}^{\infty} (\psi(z_k) - \psi(z_{k+1})) F_k(z)$$

(the functions  $F_k(z)$  are from Lemma 2, and  $\alpha_0 = \lim_{k \rightarrow \infty} \psi(z_k)$ ) such that:

1.  $f \in V_A$ .
  2.  $f(z_n) = \psi(z_n)$  ( $n = 1, 2, \dots$ ).
- II. If  $p > 2$ , then the description of  $I(l_A^p)$  is difficult already for the reason that the structure of the sets  $T = \{z_k\}_{k=1}^{\infty} \subset U$  on which functions of the class  $l_A^p$  ( $p > 2$ ) can vanish has not been studied. We prove in this direction the following theorem.

**Theorem 3.** There exists a function  $G \neq 0$  such that  $G \in \bigcap_{p>2} l_A^p$ ,

$$G(z_k) = 0, \quad |z_k| < 1 \quad (k = 1, 2, \dots)$$

and

$$\sum_{k=1}^{\infty} (1 - |z_k|) = +\infty.$$

For the proof of the theorem we shall need

**Lemma 3.** Let

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \varphi \in l_A^1; \quad \psi(z) = \sum_{k=0}^{\infty} b_k z^k, \quad \psi \in l_A^p \quad (p \geq 1).$$

Then for any  $\varepsilon > 0$  there exists  $N = N(\varepsilon, \varphi)$  such that, for  $n > N$ ,

$$\|\varphi\psi_n\|_p \leq \|\varphi\|_p \|\psi\|_p (1 + \varepsilon) \quad (p \geq 1),$$

where  $\psi_n(z) = \psi(z^n)$ .

**Proof.** Take an arbitrary  $\varepsilon > 0$ . Let  $N = N(\varepsilon, \varphi)$  be such that

$$\sum_{k=N}^{\infty} |a_k| < \varepsilon \|\varphi\|_p,$$

and let

$$S_n(z) = \sum_{k=0}^{n-1} a_k z^k, \quad R_n(z) = \sum_{k=n}^{\infty} a_k z^k.$$

Then, for  $n < N$ , we have

$$\|\varphi\psi_n\|_p = \|S_n\psi_n + R_n\psi_n\|_p \leq \|S_n\psi_n\|_p + \|R_n\psi_n\|_p;$$

since  $\|S_n\psi_n\|_p = \|S_n\|_p \|\psi_n\|_p$  for every  $n$ , and  $\|\Phi F\|_p \leq \|\Phi\|_1 \|F\|_p$  for all  $\Phi \in l_A^1$  and  $F \in l_A^p$  ( $p \geq 1$ ), it follows that

$$\|\varphi\psi_n\|_p \leq \|S_n\|_p \|\psi_n\|_p + \|R_n\|_1 \|\psi_n\|_p < \|\varphi\|_p \|\psi_n\|_p + \varepsilon \|\varphi\|_p \|\psi_n\|_p = \|\varphi\|_p \|\psi\|_p (1 + \varepsilon).$$

The lemma is proved.

**Proof of the theorem.** For the proof of the theorem we shall need the following obvious assertions:

$$xe^{-x} \leq \frac{2}{l} e^{-x/2} \quad \text{for all } x \in [0, +\infty), \quad (1)$$

$$\text{the function } y = \frac{x}{1 - e^{-x}} \text{ is increasing on } (0, +\infty). \quad (2)$$

Let

$$g_n(z) = 1 - \frac{(1 - e^{-2/n})^2}{(1 - ze^{-1/n})^2} = 1 - (1 - e^{-2/n})^2 - (1 - e^{-2/n})^2 \sum_{k=1}^{\infty} (k+1) e^{-k/n} z^k$$

$$(n = 1, 2, \dots).$$

We have the estimates:

$$(n = 1, 2, \dots).$$

$$(1 - e^{-2/n})^2 (k+1) e^{-k/n} < \left(\frac{2}{n}\right)^2 (2k) e^{-k/n} = \frac{8}{n} \left(\frac{k}{n}\right) e^{-k/n} < \frac{8}{n} e^{-k/2n} \quad (n, k = 1, 2, \dots) \quad (\text{see (1)});$$

$$\|g_n\|_p < \left(1 + \sum_{k=1}^{\infty} \left(\frac{8}{n}\right)^p (e^{-p/2n})^k\right)^{1/p} < \left(1 + \left(\frac{8}{n}\right)^p \frac{1}{1 - e^{-p/2n}}\right)^{1/p},$$

and, applying (2) ( $x = \frac{p}{2n}$ ), we obtain

$$\|g_n\|_p < \left(1 + \left(\frac{8}{n}\right)^p \frac{n}{1 - e^{-p/2}}\right)^{1/p} = \left(1 + \frac{B_p}{n^{p-1}}\right)^{1/p} \quad (p \geq 1; n = 1, 2, \dots), \quad (3)$$

where

$$B_p = \frac{8^p}{1 - e^{-p/2}}.$$

Consider a sequence of positive numbers  $\{\varepsilon_n\}_{n=1}^{\infty}$  such that

$$\sum_{k=1}^{\infty} \varepsilon_k < +\infty.$$

We construct by induction a sequence of natural numbers  $\{\lambda_n\}_{n=1}^{\infty}$  satisfying the condition

$$\|G_n\|_p \leq \|G_{n-1}\|_p \cdot \|g_n\|_p \cdot (1 + \varepsilon_n) \quad (p \geq 1; n = 1, 2, \dots), \quad (4)$$

where

$$G_0 \equiv 1, \quad G_n(z) = \prod_{k=1}^n g_k(z^{\lambda_k}) \quad (n = 1, 2, \dots).$$

Set  $\lambda_1 = 1$ . Suppose the numbers  $\lambda_1, \dots, \lambda_{n-1}$  have been constructed. The corresponding function  $G_{n-1} \in l_A^1$ . Applying the lemma to  $G_{n-1}, g_n, \varepsilon_n$ , we find  $\lambda_n$  such that

$$\|G_n\|_p \leq \|G_{n-1}\|_p \cdot \|g_n\|_p \cdot (1 + \varepsilon_n).$$

The infinite product

$$G(z) = \prod_{k=1}^{\infty} g_k(z^{\lambda_k})$$

converges uniformly in every disk  $|z| \leq r < 1$ . Consequently,  $G(z)$  is a function analytic in the disk  $|z| < 1$ ,  $G(z) \neq 0$

$$\left( 0 < G(0) = \prod_{n=1}^{\infty} \left( 1 - (1 - e^{-2/n})^2 \right) < +\infty \right).$$

For any  $p > 2$  and any  $n = 1, 2, \dots$ , applying (4), (3), we obtain

$$\begin{aligned} \|G_n\|_p &\leq \|G_{n-1}\|_p \|g_n\|_p (1 + \varepsilon_n) \leq \dots \leq \prod_{k=1}^n (\|g_k\|_p (1 + \varepsilon_k)) < \\ &< \prod_{k=1}^{\infty} \left( 1 + \frac{B_p}{k^{p-1}} \right)^{1/p} \prod_{k=1}^{\infty} (1 + \varepsilon_k) = D_p < +\infty. \end{aligned} \quad (5)$$

From the fact that  $G_n(z) \rightarrow G(z)$  uniformly in every disk  $|z| \leq r < 1$ , and from (5) it follows that

$$\|G\|_p \leq D_p < +\infty \quad (p > 2).$$

Since  $g_n(e^{-1/n}) = 0$  ( $n = 1, 2, \dots$ ), the numbers

$$z_{n,j} = \zeta_j e^{-1/n\lambda_n} \quad (j = 1, \dots, \lambda_n; n = 1, 2, \dots),$$

where  $\zeta_j$  are the roots of the equation  $\zeta^{\lambda_n} - 1 = 0$  ( $j = 1, \dots, \lambda_n$ ), are zeros of the function  $G(z)$ . Consider the sum

$$\sigma = \sum_{n=1}^{\infty} \sum_{j=1}^{\lambda_n} (1 - |z_{n,j}|) = \sum_{n=1}^{\infty} \lambda_n (1 - e^{-1/n\lambda_n});$$

since

$$\lambda_n (1 - e^{-1/n\lambda_n}) \sim_{n \rightarrow \infty} \frac{1}{n},$$

we have  $\sigma = +\infty$ . Thus,  $G(z)$  satisfies the conditions of the theorem.

**Remark 1.** The assertion of the theorem ceases to be true for  $G \in l_A^2 = H^2$  (see (5)).

**Remark 2.** Let  $G \in l_A^p$  ( $p > 2$ ),  $G \not\equiv 0$ , and  $G(z_k) = 0$ ,  $|z_k| < 1$  ( $k = 1, 2, \dots$ ). Then, since for all  $z \in U$

$$|G(z)| < \frac{M}{1 - |z|},$$

where  $M$  does not depend on  $z$ , it follows (see (8)) that

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{1+\varepsilon} < +\infty$$

for every  $\varepsilon > 0$ .

**Remark 3.** Let  $\{z_k\}_{k=1}^{\infty}$  be a sequence of numbers ( $|z_k| < 1$ ,  $k = 1, 2, \dots$ ) satisfying condition (\*) of item I. In order that there exist a function  $G \in l_A^p$  ( $p > 2$ ) such that  $G \not\equiv 0$  and  $G(z_k) = 0$  ( $k = 1, 2, \dots$ ), it is necessary and sufficient that (see (8))

$$\sum_{k=1}^{\infty} (1 - |z_k|) < +\infty.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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