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Abstract

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ON THE REPRESENTATION OF ARBITRARY FUNCTIONS BY DIRICHLET SERIES

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MATHEMATICS

We consider Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \qquad \lambda_1 \neq 0, \quad |\lambda_n| \uparrow \infty, \tag{1}$$

with, in general, complex exponents λ_n, which satisfy the condition

$$\lim_{n\to\infty} \ln n/\lambda_n = 0.$$

Under this condition, as is known, the open domain of convergence of the series coincides with the open domain of absolute convergence of the series. Let us pose the question: does there exist a domain D in which an arbitrary function $f(z)$, analytic in D or $\overline{D}$, for example $f(z) \equiv 1$, could be expanded in the series (1)?

If the λ_n are located on one or two rays, then the answer to this question is negative. Indeed, suppose first that the λ_n are located on one ray, for example, all $\arg\lambda_n = 0$. Assume that in D there is an expansion (1). Then the series (1) converges in some half-plane $\operatorname{Re} z < a$ containing the domain D, and the sum of the series tends to zero as $\operatorname{Re} z \to -\infty$. Consequently, the function $f(z) \equiv 1$ cannot be represented by the series (1). Suppose now that the λ_n are located on two rays, for example $\arg\lambda'_n = 0$, $\arg\lambda''_n = \varphi_0$, $0 < \varphi_0 \leq \pi$, $\{\lambda'_n\} \cup \{\lambda''_n\} = \{\lambda_n\}$. Suppose that in some domain D the expansion (1) is valid for $f(z) \equiv 1$. In the domain D, since the series converges absolutely, we have

$$1 = \sum_{n=1}^{\infty} a_n e^{\lambda_n z} = \sum a'_n e^{\lambda'_n z} + \sum a''_n e^{\lambda''_n z} = \Sigma_1 + \Sigma_2. \tag{2}$$

The series Σ_1 converges in the half-plane $\operatorname{Re} z < a$, containing the domain D; the series Σ_2 converges in the half-plane $\operatorname{Re}(ze^{i\varphi_0}) < b$, also containing the domain D. Both converge simultaneously in the common part G of these half-planes. For $\varphi_0 < \pi$ the domain G is an angle; along the bisector of this angle, when $z \to \infty$, the sum of the series $f(z) \equiv 1$ must tend to zero, which is impossible. For $\varphi_0 = \pi$ the domain G is a vertical strip. From equality (2) it follows that in this case the functions $f_1(z)$ and $f_2(z)$—the sums, respectively, of the series Σ_1 and Σ_2—are regular in the whole plane and bounded in modulus. Therefore $f_1(z)$ and $f_2(z)$ are constants: $f_1(z) = \alpha,\ f_2(z) = \beta$, where, since $\alpha + \beta = 1$, one of the numbers α and β is nonzero. We arrive at the preceding case. Consequently, in the case under consideration as well, $f(z) \equiv 1$ cannot have a representation (1).

The purpose of this article is to note that for some $\{\lambda_n\}$ one can indicate a domain D in which an arbitrary function $f(z)$, analytic in $\overline{D}$, admits an expansion (1). In particular, such a domain D can be indicated in the case when the λ_n are chosen in a definite way on three rays. We formulate the result obtained in this direction.

Let $L(\lambda)$ be an entire function of exponential type, and let $h(\varphi)$ be its growth indicator. Suppose that $h(\varphi) > 0,\ 0 \le \varphi < 2\pi$, and that the following condition is satisfied: there is a system of circles $|\lambda| = r_k,\ r_k \uparrow \infty$, such that

$$\ln|L(re^{i\varphi})| > [h(\varphi) - \varepsilon]r, \qquad r = r_k, \quad k > K(\varepsilon).$$

Denote by $\gamma(t)$ the function associated with $L(\lambda)$ in the sense of Borel. Let D be the smallest convex closed set containing all singularities of the function $\gamma(t)$; D is the set of interior points (the open part) of $\overline{D}$. From the condition $h(\varphi) > 0,\ 0 \le \varphi < 2\pi$, it follows that D is not empty; it contains, for example, the origin. Take an arbitrary function $f(z)$, analytic on the closed set $\overline{D}$. Let C be a convex closed contour satisfying the following conditions: the function $f(z)$ is regular on C and inside C, and the set $\overline{D}$ lies inside C. Put

$$\omega(\mu) = \frac{1}{2\pi i}\int_C \left[\int_0^\xi f(\xi - \eta)e^{\mu\eta}\,d\eta\right]\gamma(\xi)\,d\xi.$$

We assume that in the inner integral the variable η varies from 0 to ξ along a rectilinear segment. Under this condition the point $(\xi - \eta)$ does not go outside the contour C. Let $\lambda_1, \lambda_2, \ldots$ be the distinct zeros of the function $L(\lambda)$, arranged in nondecreasing order of their moduli, and let $p_1, p_2, \ldots$ be their respective multiplicities. Put further

$$P_\nu(z)e^{\lambda_\nu z} = \frac{1}{2\pi i}\int_{C_\nu} \frac{\omega(\mu)e^{\mu z}}{L(\mu)}\,d\mu \qquad (\nu = 1, 2, \ldots).$$

Here C_ν is a closed contour inside which lies λ_ν and there are no other zeros of the function $L(\lambda)$; $P_\nu(z)$ is a polynomial of degree $< p_\nu$.

Theorem. In the domain D the representation

$$f(z) = \sum_{k=1}^{\infty} f_k(z), \qquad f_1(z) = \sum_{|\lambda_\nu|<r_1} P_\nu(z) e^{\lambda_\nu z},$$

$$f_k(z) = \sum_{r_{k-1}<|\lambda_\nu|<r_k} P_\nu(z) e^{\lambda_\nu z} \qquad (k \geq 2), \tag{3}$$

holds, the series converging absolutely and uniformly inside D.

In the case when the zeros λ_n are all simple and in each annulus $r_{k-1} < |\lambda| < r_k$ there is only one zero, representation (3) takes the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \qquad a_n = \frac{\omega(\lambda_n)}{L'(\lambda_n)}, \qquad z \in D. \tag{4}$$

We indicate two simple examples where representation (4) holds. In the first example,

$$L(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^3}{n^3}\right), \qquad \{\lambda_n\} = \bigcup_{k=0}^{2} \{n e^{2k\pi i/3}\}$$

and the domain D is a regular triangle with vertices at the points $a e^{\pi i/3}$, $-a$, $a e^{-\pi i/3}$, $a = (2\sqrt{3}/3)\pi$. Here the λ_n are situated on three rays.

In the second example

$$L(z) = \sin k_1 z \, \sin i k_2 z / z^2, \qquad k_1 > 0, \qquad k_2 > 0,$$

$$\{\lambda_n\} = \{\pm k\pi/k_1\} \cup \{\pm k\pi i/k_2\}.$$

and the domain D is the rectangle $|\operatorname{Re} z| < k_2$, $|\operatorname{Im} z| < k_1$. Here the λ_n are located on four rays.

Let us note that the representation of the function $f(z)$ in the above-mentioned domain D by a Dirichlet series with exponents λ_n is not unique. Indeed, suppose in (4), for example, that $a_1 \neq 0$. Put $L_1(z) = (z - \lambda_1)^{-1} L(z)$. The function $L_1(z)$ has, like $L(z)$, all the necessary properties, but $L_1(\lambda_1) \neq 0$. Therefore in the domain D we have

$$f(z) = \sum_{n=2}^{\infty} b_n e^{\lambda_n z}.$$

The absence of uniqueness leads to the following conclusion: if, concerning a Dirichlet series with the exponents λ_n under consideration, it is known only that it converges in the domain D or in some part of it, then in principle it is impossible to indicate a formula for determining the coefficients of this series.

Let us note one more circumstance. The representation (3) cannot in general hold in a domain $G \supset \overline{D}$. Indeed, it is not difficult to show that if the series (3) converges uniformly inside some domain $G \supset \overline{D}$, then the function $f(z)$ must satisfy, in some neighborhood of the origin, the equation

$$M(f) = \frac{1}{2\pi i} \int_C \gamma(\xi) f(z+\xi)\, d\xi = 0,$$

But not every, even entire, function satisfies this equation; for example, if $\lambda \neq \lambda_n$ $(n = 1, 2, ...)$, then $M(e^{\lambda z}) = e^{\lambda z} L(\lambda) \neq 0$.

Let us indicate one possible application of the theorem. Suppose it is required to find at least one solution $F(z)$ of the nonhomogeneous equation

$$\frac{1}{2\pi i} \int_{C_1} \gamma_1(z+t) F(t)\, dt = f(z) \tag{5}$$

with a known right-hand side $f(z)$. Here $\gamma_1(u)$ is regular outside some convex closed set G. The closed contour C_1 encloses the set G. Suppose that the function $f(z)$ is regular in the convex domain $\overline{D}$ and in D is represented by the series (4). A formal solution of equation (5) will be

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n e^{\lambda_n z}}{L_1(\lambda_n)}, \qquad L_1(\mu) = \frac{1}{2\pi i} \int_{C_1} \gamma_1(t) e^{\mu t}\, dt. \tag{6}$$

The question reduces to where the series (6) will converge. Consider, for example, the case when D is a rectangle: $|\operatorname{Im} z| < k_1$, $|\operatorname{Re} z| < k_2$. In this case, as we saw above, $f(z)$ can indeed be represented by the series (4), with $\{\lambda_n\} = \{\pm n\pi/k_1\} \cup \{\pm n\pi i/k_2\}$. Suppose that, whatever $\varepsilon > 0$ may be, for sufficiently large n the expression $|L_1(\lambda_n)|$ is greater than $\exp[(\alpha - \varepsilon)|\lambda_n|]$ for $\lambda_n \in \{\pm n\pi/k_1\}$ and greater than $\exp[(\beta - \varepsilon)|\lambda_n|]$ for $\lambda_n \in \{\pm n\pi i/k_2\}$, where $\alpha \geq 0$, $\beta \geq 0$. Then the series (6) will converge uniformly inside the rectangle $D_1 :\ |\operatorname{Im} z| < k_1 + \beta$, $|\operatorname{Re} z| < k_2 + \alpha$. If it is additionally assumed that $|L_1(z)|$, for sufficiently large $|z|$, is less than $\exp[(\alpha+\varepsilon)|z|]$ on the real axis and less than $\exp[(\beta+\varepsilon)|z|]$ on the imaginary axis, then one may assert that the function (6), analytic in the rectangle D_1, is a solution of equation (5) in the rectangle D.

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