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Abstract

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ASYMPTOTIC CALCULATION OF CERTAIN ROTATIONAL MOTIONS IN THE RESONANT CASE

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§ 1. **Statement of the problem.** In papers ([5]–[7]) the question was considered of the existence and stability of stationary resonant regimes of nonlinear systems with one degree of freedom of the form:

$$\begin{gathered}\frac{d}{dt}\left[m(x)\dot{y}\right] + Q(x,y) = \varepsilon f(x,y,\dot{y},\vartheta,\varepsilon),\\ \dot{x} = \varepsilon X(x,y,\dot{y},\vartheta,\varepsilon),\\ \dot{\vartheta} = \nu(x) + \varepsilon\Theta(x,y,\dot{y},\vartheta,\varepsilon),\end{gathered} \tag{1}$$

where y is a one-dimensional coordinate; $x = \{x_1, \dots, x_n\}$ is a set of slowly varying parameters; Q is a potential force causing the motion; εf, εX, $\varepsilon\Theta$ are small nonlinear perturbations, periodically dependent on ϑ; $\varepsilon > 0$ is a small parameter; ϑ is the phase of the external perturbing force. Resonant regimes of certain systems that are special cases of (1) were considered in ([1]–[4]). In the present work, using the methods developed in ([5]), we investigate stationary resonant rotational regimes of nonlinear systems of the form

$$\begin{gathered}\dot{y} = G(x,y,p) + \varepsilon g(x,y,p,\vartheta,\varepsilon),\\ \dot{p} = F(x,y,p) + \varepsilon f(x,y,p,\vartheta,\varepsilon),\\ \dot{x} = \varepsilon X(x,y,p,\vartheta,\varepsilon),\\ \dot{\vartheta} = \nu(x) + \varepsilon\Theta(x,y,p,\vartheta,\varepsilon).\end{gathered} \tag{2}$$

System (1) is a special case of (2). For $\varepsilon = 0$, system (1) becomes an unperturbed canonical system. In this article, no assumption is made concerning a Hamiltonian form of the unperturbed system for the relatively general system of type (2). All functions in (2) are assumed periodic in y and ϑ with period

2π. Our task is to find approximations in ε for the coordinates of stationary resonant regimes and to obtain certain sufficient conditions for the stability of these regimes. Some resonant problems were considered in ([4]) by a method different from that applied here.

§ 2. **Main results.** Suppose that an integral of the degenerate system (the system into which (2) passes for $\varepsilon = 0$) is known:

$$p = P(x, y, C) \tag{3}$$

(the integral (3) is assumed to be a periodic function of y). Then we can reduce system (2) to the form

$$\begin{aligned} \dot{C} &= \varepsilon A(x, y, C, \vartheta, \varepsilon), \\ \dot{x} &= \varepsilon X(x, y, P(x, y, C), \vartheta, \varepsilon), \\ \dot{\psi} &= \omega(C, x) + \varepsilon \Psi(x, y, C, \vartheta, \varepsilon), \\ \dot{\vartheta} &= \nu(x) + \varepsilon \Theta(x, y, P(x, y, C), \vartheta, \varepsilon), \end{aligned} \tag{4}$$

where the following notation has been introduced:

$$A = \Big[-\frac{\partial P}{\partial y} g(x, y, P(x, y, C), \vartheta, \varepsilon) +$$

$$+ f(x, y, P(x, y, C), \vartheta, \varepsilon) - \frac{\partial P}{\partial x} X(x, y, P(x, y, C), \vartheta, \varepsilon) \Big] \left(\frac{\partial P}{\partial C} \right)^{-1},$$

$$\Psi = \frac{2\pi}{T} \left\{ \frac{1}{G} g(x, y, P(x, y, C), \vartheta, \varepsilon) - K \left[\frac{1}{G} \frac{\partial G}{\partial p} \frac{\partial P}{\partial C} \right] A(x, y, C, \vartheta, \varepsilon) - \right.$$

$$\left. - K \left[\frac{1}{G} \left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial p} \frac{\partial P}{\partial p} \right) \right] X(x, y, P(x, y, C), \vartheta, \varepsilon) \right\},$$

$$T = \int_{y_0}^{y_0 + 2\pi} \frac{dy}{G(x, y, P(x, y, C))}$$

is the period of rotation; $\omega = \dfrac{2\pi}{T}$ is the frequency of rotation; $\nu(x)$ is the frequency of the perturbing factors; K is an operator of the form

$$K[\varphi] = \int_{y_0}^{y} \frac{\varphi \, dy}{G} - \frac{1}{T} \int_{y_0}^{y_0 + 2\pi} \frac{\varphi \, dy}{G} \int_{y_0}^{y} \frac{dy}{G}.$$

Let us note that, in the special case for systems of the form (1), the role of the variable C may be played by the rotational energy E (see (6)).

We shall assume that resonance occurs in the system, determined by the integers p and q, i.e., that for certain values of the variables C and x the equality $p\omega = q\nu$ holds. In system (4) we pass from the variables ψ, ϑ to the variables $\varphi = \vartheta - \frac{p}{q}\psi,\ \beta = \frac{1}{q}\psi$:

$$\dot{C} = \varepsilon A(x, y, C, \varphi + p\beta, \varepsilon),$$

$$\dot{x} = \varepsilon X(x, y, P(x, y, C), \varphi + p\beta, \varepsilon), \tag{5}$$

$$\dot{\varphi} = \lambda(C, x) + \varepsilon \Phi(x, y, C, \varphi + p\beta, \varepsilon),$$

$$\dot{\beta} = \Omega(C, x) + \varepsilon B(x, y, C, \varphi + p\beta, \varepsilon),$$

where the following notation has been introduced: $\lambda = \nu - \frac{p}{q}\omega,\ \Omega = \frac{1}{q}\omega,\ \Phi = \Theta - \frac{p}{q}\Psi,\ B = \frac{1}{q}\Psi.$

Following (5), we introduce the mean values of the functions entering the right-hand sides of (5):

$$A_1 = (C_0, \varphi, x_0) =$$

$$= \frac{1}{qT(C_0, x_0)} \int_{y_0}^{y_0+2\pi q} A\left(x_0, y, C_0, \varphi + p\beta_0 + \nu_0 \int_{y_0}^{y} \frac{dy}{G(x_0, y, P)}\right) \frac{dy}{G(x_0, y, P(x_0, y, C_0))}. \tag{6}$$

The mean values of the other functions are introduced analogously to (6).

Applying to system (5) the scheme for investigating resonance regimes developed in ([5]), we find the equations for determining the resonance values of the parameters $X_0,\ C_0$ and the detuning φ_0:

$$A_1(C_0, x_0, \varphi_0) = 0, \qquad X_1(C_0, x_0, \varphi_0) = 0, \qquad \lambda(C_0, x_0) = 0. \tag{7}$$

The first-approximation corrections in ε are determined from the equations

$$\frac{\partial A_1}{\partial C_0}\delta C + \frac{\partial A_1}{\partial x_0}\delta x + \frac{\partial A_1}{\partial x_0}\delta\varphi = 0,$$

$$\frac{\partial X_1}{\partial C_0}\delta C + \frac{\partial X_1}{\partial x_0}\delta x + \frac{\partial X_1}{\partial \varphi_0}\delta\varphi = 0, \tag{8}$$

$$\frac{\partial \lambda}{\partial D_0}\delta C + \frac{\partial \lambda}{\partial x_0}\delta x + \Phi_1(C_0, x_0, \varphi_0) = 0.$$

By the stability of the stationary regimes found from (7)–(8) is understood the following: if certain stability conditions are satisfied (see ([5–7])), then for arbitrarily large $T > 0$ and arbitrarily small $\xi > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ there exists an $\eta(\varepsilon)$ such that the inequality $\max|C(t) - C_0,\ x(t) - x_0,\ \varphi(t) - \varphi_0| < \xi$ is satisfied for all

$t_0 \leq t \leq T$, if the inequality $\max|C(t) - C_0,\ x(t) - x_0,\ \varphi(t) - \varphi_0| < \eta$ is satisfied. The principal condition for stability is the requirement that the real parts of all roots of the characteristic equation $\mathrm{Det}(B - kE) = 0$ be negative, where E is the identity matrix, and the matrix B has the form

$$B = \begin{pmatrix} \varepsilon\frac{\partial A_1}{\partial C_0} & \varepsilon\frac{\partial A_1}{\partial x_0} & \varepsilon\frac{\partial A_1}{\partial \varphi_0} \\ \varepsilon\frac{\partial X_1}{\partial C_0} & \varepsilon\frac{\partial X_1}{\partial x_0} & \varepsilon\frac{\partial X_1}{\partial \varphi_0} \\ \frac{\partial \lambda}{\partial C_0} + \varepsilon\left(\frac{\partial \Phi_1}{\partial C_0} + \frac{\partial^2 \lambda}{\partial C_0 \partial x_0}\delta x + \frac{\partial^2 \lambda}{\partial C_0^2}\delta C\right) & \frac{\partial \lambda}{\partial x_0} + \varepsilon\left(\frac{\partial \Phi_1}{\partial x_0} + \frac{\partial^2 \lambda}{\partial x_0^2}\delta x + \frac{\partial^2 \lambda}{\partial x_0 \partial C_0}\delta C\right) & \varepsilon\frac{\partial \Phi_1}{\partial \varphi_0} \end{pmatrix}.$$

Let us note that, by means of this method, asymptotic approximations for stationary resonance regimes can be computed with any degree of accuracy.

§ 3. **Special cases.** Consider the nonlinear system

$$\begin{gathered} \frac{d}{dt}\left[m(x)\dot{y}\right] + F(x, y, \dot{y}) = \varepsilon f(x, y, \dot{y}, \vartheta, \varepsilon), \\ \dot{x} = \varepsilon X(x, y, \dot{y}, \vartheta, \varepsilon), \\ \dot{\vartheta} = \nu(x) + \varepsilon\Theta(x, y, \dot{y}, \vartheta, \varepsilon), \end{gathered} \tag{9}$$

which is a system of type (2). Certain special cases of systems (9) were studied in (3). All the results of § 2 are applicable to (9).

In a number of cases important for applications, finding integrals of the form (3) is substantially simplified. Thus, if the force F has the form $Q(x, y) + R(x, y)\dot{y} + S(x, y)\dot{y}^2$, then, in order to obtain the integral (3), one must solve an

Abel equation of the second kind. If $F = Q(x,y)R(x,\dot{y})$, then for the integral (3) one can obtain the expression

$$\int \frac{p\,dp}{R(x,p/m)} = -m \int Q(x,y)\,dy + C.$$

If $F = Q(x,y) + R(x,y)\dot{y}^2$, then for (3) the formula

$$p = \exp\left\{-\frac{1}{m}\int R\,dy\right\}\left(C - 2m\int Q \exp\left\{\frac{2}{m}\int R\,dy\right\}dy\right)^{1/2}.$$

is valid. Finally, if $F = R(x,y)\dot{y}^2 + S(x,y)\dot{y}^{2\alpha}$, then (3) has the form

$$p = \exp\left\{-\frac{1}{m}\int R\,dy\right\}\left(C + \frac{2(\alpha-1)}{m^{2\alpha-1}}\int S \exp\left\{\frac{2(1-\alpha)}{m}\int R\,dy\right\}dy\right)^{1/2(1-\alpha)}.$$

In all the cases listed, the results of § 2 make it possible to obtain simple quadrature formulas for asymptotic approximations of stationary resonant rotational regimes.

In conclusion, let us note that oscillatory regimes of system (2), as well as cases in which the perturbations do not depend explicitly on time, may be studied in a similar way.

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Note: Figure translations are in progress. See original paper for figures.