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Abstract

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ON THE SPECTRAL EXPANSION OF ARBITRARY SELF-ADJOINT OPERATORS IN EIGENFUNCTIONALS

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Mathematics

1. It is well known that if H is a separable Hilbert space, and A is a self-adjoint operator acting in it with simple and purely continuous spectrum, then there exists a continuous nondecreasing function $\rho(\lambda)$ on the real axis and an isometric mapping V of the space H onto the Hilbert space L^2_ρ of complex functions square-integrable with respect to the measure $\rho(\lambda)$, under which the operator A corresponds to the operator of multiplication by the independent variable. Moreover, if g is the so-called generating element, then the following relations hold:

$$(f_1, f_2) = \int_{-\infty}^{+\infty} F_1(\lambda)\overline{F_2(\lambda)}\, d\rho(\lambda), \qquad f_1, f_2 \in H; \tag{1}$$

$$Af = \int_{-\infty}^{+\infty} \lambda F(\lambda)\, d\mathcal{E}_\lambda g, \qquad f \in D_A, \tag{2}$$

where D_A is the domain of definition of the operator A; $\mathcal{E}_\lambda$ is the corresponding resolution of the identity; $F(\lambda) = Vf$; $\rho(\lambda) = \|\mathcal{E}_\lambda g\|^2$.

An analogous spectral expansion of arbitrary self-adjoint operators is also known, which, in the presence of continuous spectrum, is carried out by means of so-called differential solutions, without indicating the eigenfunctions themselves with respect to which the expansion is to be performed. Therefore, in particular, the spectral theory of differential operators continued to develop intensively ([1–13]) even after the establishment of the fundamental result contained in the representation (1), (2). It should be noted that in most works the passage from relations (1), (2) to expansions in eigenfunctions is effected by differentiating the spectral family $\mathcal{E}_\lambda g$ with respect to the measure $\rho(\lambda)$, and that the first general result, including differential operators of arbitrary type, was contained in the work ([11]), where the method applied also made it possible to prove that the eigenfunctionals obtained under the aforementioned differentiation are generalized functions in the sense of Sobolev–Schwartz.

In this note, by developing the method indicated by us earlier, a universal procedure is described that makes it possible to construct a complete system of eigenelements and eigenfunctionals for arbitrary self-adjoint operators, without restrictions on the spectral family or on the structure of the Hilbert space.

2. Let K_z be the resolvent of the operator A mentioned in Section 1; $z = \lambda + i\tau$, $\tau > 0$; g for the time being is an arbitrary fixed element of H with norm different from zero. Form the expression

$$T_z(\varphi) = \frac{1}{2\pi i \rho'(z)}((R_z - R_{\bar{z}})\varphi, g), \tag{3}$$

where $\rho'(z) = \frac{\tau}{\pi}\|R_z g\|^2$; then it is easy to verify that

$$T_z(\varphi) = \frac{(R_z\varphi, R_z g)}{\|R_z g\|^2}, \tag{3*}$$

and the following lemma is valid.

Lemma 1. For every non-real z, $T_z(\varphi)$ is a distributive and bounded functional in H and, for all $\varphi \in D_A$, satisfies the identity

$$T_z(A\varphi) = \lambda T_z(\varphi) + \frac{1}{\|R_z g\|^2}\left(\varphi, \frac{R_z + \overline{R}_z}{2}\, g\right). \tag{4}$$

Definition. A real number λ_0 is called a **generalized eigenvalue of the self-adjoint operator** A, if there exists an element $g_0 \in H$ such that

$$\liminf_{\tau \to +0} \frac{\tau}{\pi}\|R_{\lambda_0 + i\tau} g_0\|^2 > 0. \tag{5}$$

The totality of all generalized eigenvalues is called the **kernel of the spectrum of the operator** A, or its **essential spectrum**, which will be denoted by $\mathrm{Si}(A)$.

It is easy to see that the kernel of the spectrum $\mathrm{Si}(A)$ is a part of the spectrum; nevertheless the following lemma holds.

Lemma 2. The kernel of the spectrum of a self-adjoint operator A with simple and purely continuous spectrum has full spectral measure.

Indeed, it is easy to verify the inequality

$$\frac{\tau}{\pi}\|R_{\lambda+i\tau} g\|^2 = \frac{\tau}{\pi}\int_{-\infty}^{+\infty} \frac{|G(t)|^2}{(t-\lambda)^2 + \tau^2}\, d\rho(t) \geq \frac{1}{\pi}\inf |G(t)|^2 \frac{\rho(\lambda+\tau) - \rho(\lambda-\tau)}{2\tau}, \tag{6}$$

where $G(t) = Vg$; therefore, by choosing the element g in the appropriate manner, it is easy to conclude that, for all λ not belonging to the kernel of the spectrum, $\rho'(\lambda)$ exists and is equal to zero. Thus it is sufficient to prove that the set M of such points λ, where $\rho'(\lambda) = 0$, has spectral measure zero. On the other hand, by virtue of the continuity of $\rho(\lambda)$, one may conclude that the ρ-measure of the set M coincides with the Lebesgue measure of the image of the set M under the mapping ρ; therefore, invoking the known inequality ([16]), we are convinced of the validity of the lemma.

p. 3. Let $\Omega(\lambda_0, g_0)$ be the set of $\varphi \in D_A$ for which there exists

$$\lim_{\tau\to 0} \frac{(R_{\lambda_0+i\tau}\varphi, R_{\lambda_0+i\tau}g_0)}{\|R_{\lambda_0+i\tau}g_0\|^2} = T_{\lambda_0}(\varphi). \tag{7}$$

Obviously, for each fixed element $g_0 \in H$, the functional $T_{\lambda_0}(\varphi)$, defined by relation (7), is distributive on the linear manifold $\Omega(\lambda_0, g_0)$.

Lemma 3. If λ_0 belongs to the kernel of the spectrum $\mathrm{Si}(A)$, then there exists such a g_0 that $\Omega(\lambda_0, g_0)$ is everywhere dense in H and, on the linear manifold $\Omega(\lambda_0, g_0)$, the relation

$$T_{\lambda_0}(A\varphi) = \lambda_0 T_{\lambda_0}(\varphi) \tag{8}$$

holds.

Lemma 4. If the spectrum of the operator A is discrete (pure point), then, with a suitable choice* of the element g, the linear manifold $\Omega(\lambda, g)$ coincides with the whole space H for every real λ, and the functional $T_\lambda(\varphi)$ defined by relation (7) is isomorphic to the eigenvector of the operator A for those λ which are eigenvalues.

* It is enough that all coefficients of the expansion of g in the series with respect to the eigenvectors of the operator A be nonzero.

§ 4. In this section we again assume that the spectrum of the operator A is simple and purely continuous. Let C_0 be the linear space of functions continuous on the real axis and finite (vanishing outside certain compact sets), in which a sequence is convergent if it vanishes outside a fixed compact set on which it converges uniformly. Put $\Omega_A = V^{-1}C_0$; then convergence in C_0 naturally induces in Ω_A the corresponding topology and makes Ω_A into a linear space which, evidently, is everywhere dense in H and invariant with respect to the operator A, while convergence in Ω_A entails convergence in H.

Lemma 5. *With an appropriate choice of the element g, the linear manifold $\Omega(\lambda, g)$ contains the space Ω_A for all real λ, and the functionals $T_\lambda(\varphi)$ defined by the limiting relation (7) are continuous in the topology of the space Ω_A.*

Taking into account the lemmas formulated above, it is natural to call the functionals $T_\lambda(\varphi)$, obtained by the limiting relation (7) for $\lambda \in \mathrm{Si}(A)$, the **eigenfunctionals of the operator** A. Thus, one and the same relation (7) makes it possible to construct, for all λ belonging to the core of the spectrum, eigenfunctionals $T_\lambda \in \Omega_A^*$, satisfying (8) for all $\varphi \in \Omega_A$, independently of whether λ is a point of the discrete or of the continuous spectrum.

§ 5. It turns out that the functionals T_λ, constructed by means of relation (7), form a complete collection of eigenfunctionals of the self-adjoint operator A, whatever the character of its spectrum.

Let

$$\rho(z) = \rho(\lambda + i\tau) = \frac{|\tau|}{\pi} \int_{-\infty}^{\lambda} \|R_{\sigma+i\tau} g\|^2 \, d\sigma,$$

then the following is valid.

Lemma 6. *For every $\tau > 0$ and for an appropriate choice of the element g, for all φ, ψ belonging to H, the identity holds*

$$(\varphi, \psi) = \int_{-\infty}^{+\infty} T_z(\varphi)\overline{T_z(\psi)} \, d_\lambda \rho(\lambda + i\tau) +$$

$$+ \int_{-\infty}^{+\infty} \frac{(R_z\varphi, R_z\psi)(R_{zg}, R_{zg}) - (R_z\varphi, R_{zg})(R_z\psi, R_{zg})}{\|R_{zg}\|^4} \, d_\lambda \rho(\lambda + i\tau). \qquad (9)$$

It is known [17] that for any interval (α, β)

$$\lim_{\tau\to 0} \big[\rho(\beta + i\tau) - \rho(\alpha + i\tau)\big] = \rho(\beta) - \rho(\alpha).$$

At the same time, one can prove that, for fixed φ, ψ belonging to Ω_A, the integrand of the second integral in relation (9) tends to zero as $\tau \to 0$ at all points λ belonging to the core of the spectrum $\mathrm{Si}(A)$. An investigation of the nature of this convergence, based on the use of a number of known results [17] on the behavior of the resolvent in a neighborhood of the real axis, as well as a certain refinement of Lemma 3, makes it possible to carry out the limiting passage under the integral sign in relation (9), which leads to the important formula

$$(\varphi, \psi) = \int_{-\infty}^{+\infty} T_\lambda(\varphi)\overline{T_\lambda(\psi)} \, d\rho(\lambda), \qquad \varphi, \psi \in \Omega_A. \qquad (10)$$

From formula (8) it follows, in particular, that if $f \in \Omega_A$ and $T_\lambda(f) = 0$ for all $\lambda \in \mathrm{Si}(A)$, then $f = 0$, i.e. completeness.

Thus we arrive at the following theorem.

Theorem 1. *Let A be a self-adjoint operator with simple and purely continuous spectrum; then the collection of functionals $\{T_\lambda;\ \lambda \in \mathrm{Si}(A)\}$, constructed by formula (7), forms a complete system of eigenfunctionals, continuous in the* *topology of the space Ω_A, and every functional from Ω_A^*, isomorphic to an element of Ω_A, admits an expansion into a generalized Fourier-Stieltjes integral with respect to these functionals.*

§ 6. Let now A be an arbitrary self-adjoint operator; then it is known [17] that the space H can be represented as an orthogonal sum of at most a countable number of subspaces H_k invariant with respect to the operator A, in each of which the spectrum of the operator A is simple; therefore the transfer of the results formulated to this general case requires no additional considerations. We note that it is apparently expedient to classify points of the spectrum depending on the rate at which

$$\sup_{\|g\|=1} \|R_{\lambda+i\tau} g\|^2 \tag{11}$$

tends to infinity as $\tau \to 0$. In the generally accepted classification, a real point λ is regular if expression (11) does not tend to infinity at all, and belongs to the spectrum if this expression tends to infinity arbitrarily slowly. It follows from Lemma 2 that, in constructing spectral expansions, it is quite sufficient to restrict oneself to those λ for which expression (11) tends to infinity no more slowly than $1/\tau$ (it is precisely such λ that form the essential spectrum $\mathrm{Si}(A)$).

Moreover, Lemma 4 can be supplemented as follows:

Lemma 4^*. *A point λ is an eigenvalue of the operator A if and only if expression (11) is $O(1/\tau^2)$.*

On the other hand, one can prove that if the spectrum of the operator is Lebesgue, then expression (11), for almost all λ in the sense of Lebesgue measure, is $O(1/\tau)$. Apparently the converse is also true; however, we can prove this only under the additional assumption that the constant in $O(1/\tau)$, as a function of λ, is summable in the Lebesgue sense.

§ 7. Thus, the results formulated in this note show that one and the same procedure (the limiting relation (7)) makes it possible to construct a complete system of eigenfunctionals for an arbitrary self-adjoint operator, without distinguishing between points of the discrete or continuous spectrum; however, only at points of the discrete spectrum do the eigenfunctionals obtained turn out to be isomorphic to elements of H. We note that the method we propose for constructing eigenfunctionals, in contrast to the methods mentioned in § 1, does not presuppose the preliminary construction of a spectral family $\mathcal{E}_\lambda$, which proves advantageous in the study of a number of concrete operators, especially in those cases when the structure of the complete system of eigenfunctionals is known in advance.

In conclusion we note that we arrived at the indicated considerations, which concern general self-adjoint operators, as a result of our attempts to investigate qualitative properties of solutions of certain systems of the type of S. L. Sobolev;

this turned out to be equivalent to investigating spectral properties of hyperbolic operators, where the eigenfunctions are, as a rule, discontinuous functions and therefore must be treated as functionals.

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