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Abstract

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MATHEMATICS

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ON THE THEORY OF GENERALIZED SOLV- ABLE GROUPS*

(Presented by Academician A. I. Mal' tsev on 10 IX 1964)

The important role of the normalizer condition in group theory is well known. In the present note (§ 1) the concept of a weakly normal subgroup is introduced; with its help an analogue of the normalizer condition is constructed for groups close to the groups studied in ⁽¹⁾, and also for groups close to groups possessing ascending invariant series with cyclic factors (see Theorems 1 and 3).

Theorem 3 is analogous to a known result of B. I. Plotkin on the local nilpotence of an N -group (a group with the normalizer condition) ⁽²⁾. However, its proof differs essentially from B. I. Plotkin' s proof, whose result is in fact its consequence. In addition, Theorem 3 implies the supersolvability of a group with a finite number of generators that possesses an ascending invariant series with cyclic factors ⁽³⁾.

Theorems 2 and 4 become known propositions for finite nilpotent groups if, in their formulations, "uniform" (respectively "quasiuniform") is replaced by "direct" ; a ZH -series (respectively a supersolvable one) by a central series; an open weakly normal subgroup by a subgroup distinct from its normalizer; and, finally, an \mathfrak{H} -uniform series by a normal series passing through the subgroup \mathfrak{H} . (Definitions of the uniform and quasiuniform products of a ZH -series are given in ⁽¹⁾.)

In § 2 Theorem 5 is formulated, developing certain ideas from R. Brauer' s report at the Amsterdam Congress in 1954 ⁽⁴⁾. From this theorem, in view of the theorem of Feit and Thompson ⁽⁵⁾ and the results of S. N. Chernikov ⁽⁶⁾, there follows in particular a positive solution of O. Yu. Schmidt' s problem for locally finite groups, already obtained earlier by M. I. Kargapolov ⁽⁷⁾. Theorem 6 gives a sufficiently complete description of locally finite \mathfrak{S} -groups.

The incentive that prompted the author to study the questions considered in § 2 was the work of M. I. Kargapolov ⁽⁷⁾.

§ 1. Definitions. Let \mathfrak{G} be a group; \mathfrak{H} its subgroup; and A and B some elements of \mathfrak{G} . Let n_A and n_B be, respectively, the orders of the elements A and B relative to the subgroup \mathfrak{H} , i.e. the least natural numbers for which $A^{n_A} \in \mathfrak{H}$, $B^{n_B} \in \mathfrak{H}$. Define the number μ as follows: 1) $\mu = 0$ if n_A and n_B are infinite; 2) if n_A and n_B are finite, then μ is equal to the product of all

prime divisors of the greatest common divisor of the numbers n_A and n_B . In particular, if n_A and n_B are relatively prime, then $\mu = 1$. If one of the numbers n_A or n_B is finite and the other infinite, then the number μ is taken to be equal to the product of all prime divisors of that one of the numbers n_A and n_B which is finite.

The elements A and B will be called **arithmetically connected relative to the subgroup \mathfrak{H}** if, for every $H \in \mathfrak{H}$ for which the relations

$$HA = A^k H', \quad HB = B^r H'',$$

hold, where $H' \in \mathfrak{H}$, $H'' \in \mathfrak{H}$, one has

$$k \equiv r \pmod{\mu},$$

where μ is the number defined above.

* The contents of the present note were reported at the Sixth All-Union Colloquium on General Algebra in June 1964 in Minsk.

A subgroup \mathfrak{H} of a group \mathfrak{G} will be called **weakly normal** in \mathfrak{G} if any two elements A and B of \mathfrak{G} such that each of the subgroups $\{A\}$ and $\{B\}$ permutes with \mathfrak{H} are arithmetically connected relative to \mathfrak{H} . A subgroup \mathfrak{H} of a group \mathfrak{G} is **weakly normal relative to a certain element $B \in \mathfrak{G}$** if \mathfrak{H} and $\{B\}$ permute and any two elements B^r and B^s (r and s are integers) such that each of the subgroups $\{B^r\}$ and $\{B^s\}$ permutes with \mathfrak{H} are arithmetically connected relative to \mathfrak{H} . A weakly normal subgroup \mathfrak{H} of a group \mathfrak{G} will be called **open weakly normal** in \mathfrak{G} if, for $\mathfrak{H} \neq \mathfrak{G}$, in \mathfrak{G} there exists at least one cyclic subgroup not contained in \mathfrak{H} and permuting with it. A series of subgroups

$$\mathfrak{G} = \mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_2 \subseteq \dots \subseteq \mathfrak{H}_n = \mathfrak{G}$$

will be called **weakly normal** if \mathfrak{H}_i is weakly normal in \mathfrak{H}_{i+1} ($i = 0, 1, \dots, n-1$).

A series of subgroups

$$\mathfrak{H} = \mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \dots \subseteq \mathfrak{H}_n = \mathfrak{G}$$

will be called an **\mathfrak{H} -uniform series** if $\mathfrak{H}_{i+1} = \{\mathfrak{H}_i, A_{i+1}\}$ and the subgroup \mathfrak{H}_i is weakly normal relative to the element A_{i+1} . In particular, if $\mathfrak{H} = \mathfrak{G}$, then an \mathfrak{H} -uniform series will be called simply **uniform**.

Analogously one can define an ascending weakly normal series and an ascending \mathfrak{H} -uniform series.

Proposition 1. *If a group \mathfrak{G} has an (ascending) \mathfrak{H} -uniform series, then any of its subgroups containing \mathfrak{H} has an (ascending) \mathfrak{H} -uniform series.*

Proposition 2. *An ascending \mathfrak{H} -uniform series of a group \mathfrak{G} (and also any of its refinements to an ascending \mathfrak{H} -uniform series) is an ascending weakly normal series.*

Periodic groups all of whose subgroups are open weakly normal are described by the following theorem.

Theorem 1. *A periodic group \mathfrak{G} will be a group all of whose subgroups are open weakly normal in it if and only if it is an extension of a Hall abelian subgroup \mathfrak{H} , all of whose subgroups are invariant in \mathfrak{G} , by an N -group.*

The subgroup \mathfrak{H} need not be complemented in \mathfrak{G} . As indicated in paper ⁽¹⁾, a contradictory example was constructed by Yu. M. Gorchakov ⁽⁸⁾.

Theorem 2. *For finite groups the following properties are equivalent: 1) the group decomposes into the uniform product of its Sylow P -subgroups; 2) the group has a ZH -series; 3) the group in which every subgroup is open weakly normal.*

Definition. We shall call a group \mathfrak{G} an **NZ -group** if through every subgroup \mathfrak{H} of it there passes an ascending \mathfrak{H} -uniform series of the group \mathfrak{G} .

Theorem 3. *An NZ -group with a finite number of generators is supersolvable.*

Corollary 1. *A ZA -group with a finite number of generators is nilpotent ⁽²⁾.*

Corollary 2. *An N -group with a finite number of generators is nilpotent ⁽²⁾.*

Since a group possessing an ascending invariant series with cyclic factors is obviously an NZ -group, the following result, proved by Baer, follows from Theorem 3:

A group with a finite number of generators possessing an ascending invariant series with cyclic factors is supersolvable ⁽³⁾.

Theorem 4. *For finite groups the following properties are equivalent: 1) the group decomposes into the quasi-uniform product of its Sylow p -subgroups; 2) the group is supersolvable; 3) the group is an NZ -group.*

§ 2. Theorem 5. *If in a locally finite group the centralizer of some involution (element of order 2) is finite, then in it there exists a normal divisor of finite index in it not containing this involution.*

In view of the Feit–Thompson theorem ⁽⁵⁾ and results of S. N. Chernikov ⁽⁶⁾, the following propositions follow from Theorem 5:

Corollary 1. *If an infinite locally finite group is simple, then the centralizer of any involution in it is infinite.*

Corollary 2. *An infinite locally finite group all of whose proper subgroups are finite is quasicyclic (O. Yu. Schmidt' s problem for locally finite groups).*

Corollary 3. *If in a locally finite group every abelian subgroup containing involutions is finite, then the group itself is a finite extension of a locally soluble group.*

From the last proposition, in view of the results of S. N. Chernikov ⁽⁶⁾, as is not hard to see, there follows the following positive solution of a problem from the survey ⁽⁹⁾, also solved by M. I. Kargapolov ⁽⁷⁾:

If in a locally finite group all abelian subgroups are finite, then it is finite.

Definition. We shall call a group \mathfrak{G} an \bar{S} -group if every one of its proper infinite subgroups is locally nilpotent.

Theorem 6. A locally finite \bar{S} -group is either locally nilpotent or extremal.

In accordance with the survey ⁽⁹⁾, a group is called **extremal** if it is a finite extension of an abelian group satisfying the minimal condition for subgroups.

The proof of Theorem 6 follows from the lemmas given below. In those lemmas in which a group \mathfrak{G} is discussed, it will be assumed that it satisfies the hypotheses of Theorem 6.

Lemma 1. If in a locally finite group with the minimal condition for subgroups there exists a maximal subgroup which is locally nilpotent, then the whole group is extremal.

In the proof of Lemma 1 one uses Theorem 5, the results of J. Thompson ⁽¹⁰⁾, S. N. Chernikov ⁽⁶⁾, Gorenstein ⁽¹¹⁾, and the generalized Frobenius theorem for locally finite groups ^(12,13).

Lemma 2. Every locally finite p -group is extremal if it contains a maximal elementary subgroup of finite order.

Lemma 3. If in the group \mathfrak{G} there exists an infinite proper normal divisor, then Theorem 6 is valid for it.

Lemma 4. If the group \mathfrak{G} is locally soluble, then Theorem 6 is valid for it.

Proof. It is known that a locally soluble noncyclic group is not simple ⁽²⁾. Consequently, if \mathfrak{G} is infinite, then it has a proper normal divisor. Using Lemma 3, it is not hard to see that Lemma 4 need only be proved for the case when the group \mathfrak{G} has an ascending upper central series all of whose terms are finite groups. It is known that such a group is extremal ⁽¹⁴⁾.

Lemma 5. Let p be some number from $\pi(\mathfrak{G})$ ($\pi(\mathfrak{G})$ is the set of all prime divisors of the orders of elements of the group \mathfrak{G}), and suppose that in the group \mathfrak{G} there are no nonidentity invariant p -subgroups. Then, if at least one Sylow p -subgroup of the group \mathfrak{G} is extremal, all Sylow p -subgroups are extremal (for this p).

In the proof of Lemma 5, Lemma 2 is used essentially.

Lemma 6. If the group \mathfrak{G} is not locally nilpotent, then for every prime number $p \in \pi(\mathfrak{G})$ the Sylow p -subgroups of \mathfrak{G} are extremal.

Proof. Let \mathfrak{P} be a Sylow p -subgroup of \mathfrak{G} , and let \mathfrak{A} be a maximal invariant p -subgroup. In view of the hypotheses of the lemma being proved, $\mathfrak{A} \neq \mathfrak{G}$. If

\mathfrak{A} is infinite, then Lemma 6 follows from Lemma 3. Suppose that \mathfrak{A} is finite. Obviously, we may assume that \mathfrak{G} is an infinite group. The centralizer $Z_{\mathfrak{G}}(\mathfrak{A})$ of the subgroup \mathfrak{A} in \mathfrak{G} is invariant in \mathfrak{G} , has finite index in it and, consequently, is an infinite group. If $Z_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{G}$, then the validity of Lemma 6 follows from Lemma 3. Let $Z_{\mathfrak{G}}(\mathfrak{A}) \neq \mathfrak{G}$; then the factor group $\mathfrak{G}/\mathfrak{A}$ does not decompose into a direct sum ...

the product of primary Sylow subgroups. Indeed, if it decomposes into a direct product of such subgroups, then the group \mathfrak{G} also, obviously, decomposes into a direct product of primary Sylow subgroups, which contradicts the condition of Lemma 6. Consequently, it suffices to prove the lemma for the case $\mathfrak{A} = 1$. Thus, we shall assume that $\mathfrak{A} = 1$.

Suppose that in the group \mathfrak{G} there exists a Sylow p -subgroup that is not extremal; then, in view of Lemma 5, the Sylow p -subgroups are not extremal.

Let \mathfrak{H} be any finite elementary p -subgroup of \mathfrak{G} . We shall prove that if some element $A \in \mathfrak{G}$ of order not divisible by p is permutable with \mathfrak{H} , then it is elementwise permutable with it.

Indeed, the subgroup \mathfrak{H} is contained in some Sylow p -subgroup \mathfrak{P} of the group \mathfrak{G} , and the subgroup \mathfrak{P} , as noted above, is not extremal. Consequently, in view of Lemma 2, the subgroup \mathfrak{H} can be embedded in an infinite abelian p -subgroup $\bar{\mathfrak{H}} \subseteq \mathfrak{P}$. Obviously,

$$\bar{\mathfrak{H}} \triangle \mathfrak{G}_1 = \{\bar{\mathfrak{H}}, A\}.$$

Since, by assumption, $\mathfrak{A} = 1$, we have $\mathfrak{G}_1 \neq \mathfrak{G}$. The subgroup \mathfrak{G}_1 is infinite and, consequently, decomposes into a direct product of primary Sylow subgroups. Hence it obviously follows that the element A is elementwise permutable with $\bar{\mathfrak{H}}$. Further, from this it is not hard to obtain, using the conditions of the lemma being proved, and also Hall's theorem on the invariant complement of a Sylow p -subgroup in finite groups ⁽¹⁵⁾, that all elements of the group \mathfrak{G} whose orders are not divisible by p generate an invariant subgroup \mathfrak{N} not containing p -elements. The factor group $\mathfrak{G}/\mathfrak{N}$ is a p -group. Applying to the subgroup \mathfrak{N} arguments analogous to those carried out for the subgroup \mathfrak{A} at the beginning of the proof of the lemma, we arrive at a contradiction to our assumption that the rank of the subgroup \mathfrak{P} is infinite. The lemma is proved.

Since every finite group of odd order is soluble ⁽⁵⁾, Theorem 6 is easily proved by applying Lemmas 1, 4, 6 and the generalized Frobenius theorem ^(12,13).

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