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## Abstract

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# MATHEMATICS

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# ON A FAMILY OF DIFFERENTIAL GAMES OF SURVIVAL IN THE SPACE $R^n$

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A family of antagonistic games is considered, each of which is a model of pursuit in some closed convex set $S \subset R^n$. Two points, the pursuer $P$ and the evader $E$, possessing constant linear velocities $v$ and $u$ ($v > u$), move in the set $S$, while having the possibility at every moment of time to change the direction of their motion. The evader $E$ is considered caught as soon as the points $P$ and $E$ coincide. The aim of $E$ is to reach the boundary of the set $S$ before being caught by the pursuer. Before giving an exact definition of the game, let us define two sets of vector-functions.

Put $T = [0, \infty]$. Let $\mathfrak{C}$ be the set of vector-functions

$$\psi(x_1, \dots, x_n, t) = [\psi_1(x_1, \dots, x_n, t), \dots, \psi_n(x_1, \dots, x_n, t)],$$

defined on $R^n \times T$ with values in $R^n$; $\mathfrak{P}$ is the set of vector-functions

$$\varphi[\psi(x_1, \dots, x_n, t)] = \{\varphi_1[\psi(x_1, \dots, x_n, t)], \dots, \varphi_n[\psi(x_1, \dots, x_n, t)]\},$$

depending on $\psi(x, t)$, with values in $R^n$ and, for fixed $\psi$, defined on $R^n \times T$.

The following restrictions are imposed on the elements of the sets $\mathfrak{C}$ and $\mathfrak{P}$:

1. For every $\psi \in \mathfrak{C}$ there exists a sequence $0 = t_0 < t_1 < \dots < t_m = \infty$ such that the space $R^n \times T$ can be divided into a finite number of regions $R_1, \dots, R_{m-1}$, each of which is representable in the form of a Cartesian product $R_k = R^n \times [t_k, t_{k+1})$, $k = 1, 2, \dots, m-1$, and on each $R_k$ ($k = 1, 2, \dots, m-1$) the function $\psi \in \mathfrak{C}$ and the corresponding $\varphi(\psi) \in \mathfrak{P}$ are continuous together with their first partial derivatives.
2. For any $\varphi \in \mathfrak{P}$ and $\psi \in \mathfrak{C}$, the system of equations

$$\begin{aligned} \dot{x}_i &= \varphi_i[\psi(x_1, \dots, x_n, t)], \qquad & i = 1, 2, \dots, n; \\ \dot{y}_j &= \psi_j(x_1, \dots, x_n, t), \qquad & j = 1, 2, \dots, n, \end{aligned} \tag{1}$$

has a unique solution for any initial conditions $\xi, \eta \in S$.

3.

$$\varphi_1^2 + \cdots + \varphi_n^2 = v^2;$$

$$\psi_1^2 + \cdots + \psi_n^2 = u^2,$$

where $v = \text{const}$, $u = \text{const}$, $v > u$.

4. Let $\{x(t), y(t)\}$ be a solution of system (1) under the initial conditions $\xi, \eta \in S$, and let

$$t_{S_E} = \inf\{t : y(t) \in \overline{S}\},$$

$$t_{S_P} = \inf\{t : x(t) \in \overline{S}\}.$$

Then for $t > t_{S_E}$, $y(t) \in \overline{S}$, and for $t > t_{S_P}$, $x(t) \in \overline{S}$.

For any $\xi, \eta \in S$ we define a differential game of survival, which we agree to denote by $\Gamma(\xi, \eta)$. The sets of vector-functions $\mathfrak{C}$

and $\mathfrak{P}$, satisfying conditions 1-4, are the sets of pure strategies of players $P$ and $E$ in the game $\Gamma(\xi, \eta)$. To each situation $(\varphi, \psi)$, under initial conditions $\xi, \eta \in S$, there correspond uniquely two trajectories $\{x(t), y(t)\}$, which are the solution of system (1) and are called the trajectories of the pursuer $P$ and the pursued $E$.

Put

$$t_P = \min\{t : x(t) = y(t)\}.$$

In any situation the quantity $t_P$ is uniquely defined and may be equal to some finite number or to infinity.

The payoff function is defined as follows:

$$K(\xi, \eta, \varphi, \psi) = \begin{cases} +1, & t_{S_E} \geq t_P, \\ -1, & t_{S_E} < t_P, \\ 0, & t_{S_E} = \infty, \quad t_P = \infty. \end{cases} \tag{2}$$

The game $\Gamma(\xi, \eta)$ was first considered by Isaacs ([1]) under the name "the lifeline game."

**Definition.** A strategy $\varphi^{\Pi} \in \mathfrak{P}$ is called a $\Pi$-strategy if, in any situation $(\varphi^{\Pi}, \psi)$, the point $x(t)$ lies on the straight line passing through $y(t)$ and parallel to the straight line passing through $\xi$ and $\eta$, and $t_P < \infty$.

Place the origin of coordinates at the point $\eta$ and direct the coordinate unit vector $e_n$ toward the point $\xi = (\xi_1, \dots, \xi_n)$. Let $\psi$ be some strategy of $E$. Then, from the definition of a $\Pi$-strategy, it follows that in the situation $(\varphi^\Pi, \psi)$ the trajectory $x(t)$ is obtained as the solution of the system of equations

$$\dot{x}_i = \psi_i,$$

$$\dot{x}_n = -\left(v^2 - \sum_{i=1}^{n-1} \psi_i^2\right)^{1/2}, \qquad i = 1, 2, \dots, n-1,$$

with initial conditions

$$x_i(0) = 0, \quad x_n(0) = \xi_n, \quad i = 1, 2, \dots, n-1.$$

**Theorem 1.** *In the game $\Gamma(\xi, \eta)$ there exists an equilibrium situation $(\varphi^0, \psi^0)$ ([2]), in which $\varphi^0$ is a $\Pi$-strategy, while $\psi^0$ is a solution of the functional equation*

$$K(\xi, \eta, \varphi^\Pi, \psi^0) = \min_{\mathfrak{E}} K(\xi, \eta, \varphi^\Pi, \psi).$$

If additional restrictions (on angular accelerations) of the form

$$\left(\frac{d\varphi_1}{dt}\right)^2 + \dots + \left(\frac{d\varphi_n}{dt}\right)^2 \le w_P^2,$$

$$\left(\frac{d\psi_1}{dt}\right)^2 + \dots + \left(\frac{d\psi_n}{dt}\right)^2 \le w_E^2,$$

where $w_P$ and $w_E$ are certain fixed constants, are imposed on the classes of strategies of players $P$ and $E$, then Theorem 1 remains valid if $w_P \ge w_E$.

Denote $z = (\xi, \eta)$. Let $V(z)$ be the value of the game $\Gamma(z)$. At each point $z \in S \times S$, $V(z)$ may take one of two values, 0 or 1.

**Definition.** The set

$$\mathfrak{B}(S) = \{z : V(z) = 1\}$$

will be called the **winning set** of the pursuer $P$.

Since the construction of the set $\mathfrak{B}(S)$ completely describes the analysis of the family of games $\Gamma(z)$ for $z \in S \times S$, we shall undertake the study of this set. It turns out that the value of the games $\Gamma(z)$ depends on two quantities: on the set $S$ and on the ratio $\gamma = v/u$. In accordance with this, below, instead of $\Gamma(z)$, $V(z)$, $\mathfrak{B}(S)$, we shall write $\Gamma(S, \gamma, z)$, $V(S, \gamma, z)$, $\mathfrak{B}(S, \gamma)$.

**Lemma 1.** *Suppose the games $\Gamma(S_1, \gamma, z), \dots, \Gamma(S_n, \gamma, z), \dots$ and the game*

$$\Gamma\left(\bigcap_1^\infty S_n, \gamma, z\right).$$

Then

$$\mathfrak{B}\left(\bigcap_{1}^{\infty} S_n, \gamma\right) = \bigcap_{1}^{\infty} \mathfrak{B}(S_n, \gamma).$$

**Lemma 2.** The set $\mathfrak{B}(S, \gamma)$ is closed.

Let $z^0 = (\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) \in S \times S$; then by the **section** of the set $\mathfrak{B}(S, \gamma)$ at the point $\xi^0 = (\xi_1^0, \xi_2^0)$ we shall mean the set

$$\mathfrak{B}_{\xi^0}(S, \gamma) = \{z : z = (\xi_1^0, \xi_2^0, \eta_1, \eta_2) \in \mathfrak{B}(S, \gamma)\},$$

and by the section of $\mathfrak{B}(S, \gamma)$ at the point $\eta^0 = (\eta_1^0, \eta_2^0)$—the set $\mathfrak{B}_{\eta^0}(S, \gamma) = \{z : z = (\xi_1, \xi_2, \eta_1^0, \eta_2^0) \in \mathfrak{B}(S, \gamma)\}$. We shall find the form of the sets $\mathfrak{B}_\xi(S, \gamma)$ and $\mathfrak{B}_\eta(S, \gamma)$ in the case when $S$ is a closed half-plane.

$\mathfrak{B}_\xi(S)$ is given by the inequality in polar coordinates with origin at the point $\xi$

$$\frac{ruv}{v^2 - u^2} + \frac{rv^2}{v^2 - u^2} \cos\varphi - b \geq 0, \tag{3}$$

where $b$ is the distance from the point $\xi$ to the boundary.

The set $\mathfrak{B}_\eta(S, \gamma)$ is given by the inequality with origin at the point $\eta$, in polar coordinates,

$$\frac{ruv}{v^2 - u^2} + \frac{ru^2}{v^2 - u^2} \cos\varphi - c \leq 0, \tag{4}$$

where $c$ is the distance from the point $\eta$ to the boundary of the set $S$.

Let $S$ be an arbitrary convex closed set; then it is representable as a countable intersection of half-planes $S_k$, $S = \bigcap_1^\infty S_k$.

Hence, by the lemma, $\mathfrak{B}(S, \gamma) = \bigcap_1^\infty \mathfrak{B}(S_k, \gamma)$; passing to the section at the point $\xi$, we obtain $\mathfrak{B}_\xi(S, \gamma) = \bigcap_1^\infty \mathfrak{B}_\xi(S_k, \gamma)$, and passing to the section at the point $\eta$, we obtain

$$\mathfrak{B}_\eta(S, \gamma) = \bigcap_{1}^{\infty} \mathfrak{B}_\eta(S_k, \gamma).$$

Thus, the sets $\mathfrak{B}_\xi(S, \gamma)$ and $\mathfrak{B}_\eta(S, \gamma)$ can be approximated by sets of the form $\bigcap_1^N \mathfrak{B}_\xi(S_k, \gamma)$ and $\bigcap_1^N \mathfrak{B}_\eta(S_k, \gamma)$. However, these sets are sections of the winning set for an $N$-vertex polygon, which are easily obtained by combining inequalities (3) and (4) for its sides and applying Lemma 1.

The following theorem describes the change of $\mathfrak{B}(S,\gamma)$ with the change of $\gamma = v/u$ for fixed $S$.

**Theorem 2.** 1) Let $\gamma_k > \gamma,\ k = 1, 2, \dots,\ \lim_{k\to\infty} \gamma_k = \gamma$; then

$$\mathfrak{B}(S,\gamma) = \bigcap_{1}^{\infty} \mathfrak{B}(S,\gamma_k).$$

2) Let $\gamma_k \leq \gamma,\ k = 1, 2, \dots,\ \lim_{k\to\infty} \gamma_k = \gamma$; then

$$\mathfrak{B}(S,\gamma) = \bigcup_{1}^{\infty} \mathfrak{B}(S,\gamma_k).$$

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## REFERENCES CITED

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2. R. D. Luce, H. Raiffa, Games and Decisions, IL, 1961.

*Note: Figure translations are in progress. See original paper for figures.*