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Abstract

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On the Behavior at Infinity of Eigenfunctions of the Schrödinger Operator

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In this note we establish some estimates of the growth at infinity, with respect to the spatial variables, of the spectral kernel (and hence also of the eigenfunctions) of an arbitrary self-adjoint extension in $L_2(E^n)$ (E^n is n-dimensional Euclidean space) of the Schrödinger operator S

$$(S\varphi)(x) = -\sum_{j=1}^{n} \frac{\partial^2 \varphi(x)}{\partial x_j^2} + q(x)\varphi(x), \qquad \varphi \in D(S) = C_0^\infty(E^n), \quad q \in C^\infty(E^n).$$

Semiboundedness of S is not assumed. These results extend to the case of arbitrary sufficiently smooth potentials $q(x)$ the estimates previously established for semibounded $q(x)$ (see, for $n = 1$, the results of I. E. Shnol' (^{1}); for $n > 1$, the results of Yu. M. Berezanskii, G. I. Kats and A. G. Kostyuchenko (^{2}), and Yu. M. Berezanskii (^{3}), Ch. 6, § 2; estimates for the case of bounded potentials follow from the results of A. G. Kostyuchenko (^{4})).

1. Let A be some self-adjoint extension of the operator S; E_Δ^A the resolution of the identity of the operator A; $E_\Delta^A = \int_\Delta dE_\lambda^A$; Δ an arbitrary compact Borel set on the axis E^1. By a spectral kernel we mean the kernel $\psi_A(x, y; \lambda)$ participating in Parseval' s equality:

$$(E_\Delta^A f, g) = \int_\Delta \left\{ \int_{E^n} \int_{E^n} \psi_A(x, y; \lambda) f(y) \overline{g(x)} \, dx \, dy \right\} d\rho_A(\lambda). \tag{1}$$

Here $f(x), g(x) \in L_{2,0}(E^n)$, i.e. these are functions in $L_2(E^n)$ with compact support; ρ_A is the spectral measure of the operator A; $\psi_A(x, y; \lambda)$, for ρ_A-almost all λ, is an infinitely differentiable function of the variables x, y, which is a solution of the equation

$$-\sum_{j=1}^{n} \frac{\partial^2 \psi_A(x, y; \lambda)}{\partial x_j^2} + q(x)\psi_A(x, y; \lambda) = \lambda \psi_A(x, y; \lambda) \tag{2}$$

and of the analogous equation in y (see, for example, (^{3}), Ch. 6, § 2; (^{5})).

The following assertion characterizes the behavior of the spectral kernels $\psi_A(x,y;\lambda)$ as $|x|,|y|\to\infty$.

Theorem. Let A be an arbitrary self-adjoint extension in $L_2(E^n)$ of the operator S. For any $\varepsilon,\delta>0$ there exists a set of full ρ_A-measure $\Omega_{\varepsilon,\delta}\subset E^1$ (i.e. $\rho_A(E^1\setminus\Omega_{\varepsilon,\delta})=0$) such that for $\lambda\in\Omega_{\varepsilon,\delta}$ the estimate

$$\psi_A(x,y;\lambda)_{|x|^2+|y|^2\to\infty} = O\left(e^{\delta\left(\sqrt{q^\delta_{\min}(x)}+\sqrt{q^\delta_{\min}(y)}\right)}|x|^{n/2+\varepsilon}|y|^{n/2+\varepsilon}\right), \qquad q^\delta_{\min}(x) = -\min_{|\xi-x|\le\delta}\min(q(\xi),0). \tag{3}$$

It follows from the theorem that for almost all eigenfunctions $\varphi(x,\lambda)$ of the operator A

$$\varphi(x,\lambda) = O\left(e^{\delta\sqrt{q^\delta_{\min}(x)}}|x|^{n/2+\varepsilon}\right), \quad \delta,\varepsilon>0.$$

An estimate of the form (3) was first obtained by Yu. M. Berezanskii under certain restrictions on S for E^n, $n\le 3$ (see (^{3}), Ch. 6). If $q(x)\ge\text{const}$, then from (3) follows the asymptotic formula

$$\psi_A(x,y;\lambda) = \underset{|x|^2+|y|^2\to\infty}{O}\left(|x|^{n/2+\varepsilon}|y|^{n/2+\varepsilon}\right),$$

which generalizes to arbitrary E^n the result of Yu. M. Berezanskii, T. I. Kac, and A. G. Kostyuchenko (^{2}), pertaining to the case $n=3$.

The theorem is proved by means of a method generalizing the arguments of the authors mentioned above. One can verify that estimate (3) is a consequence of the following property of the operator S:

There exists a family of real continuous functions $\gamma_t(\lambda)$, $\lambda\in E^1$, $0<t<\infty$, such that: a) for any self-adjoint extension A, the $\gamma_t(A)$, $t>0$, are integral operators, and moreover $\gamma_t(A')\varphi=\gamma_t(A'')\varphi$, $\varphi\in C_0^\infty(E^n)$, for any two extensions A', A''; b) the kernels $\Gamma_t(x,y)$ of the operators $\gamma_t(A)$ are bounded if x,y range over bounded domains; $\Gamma_t(x,y)=0$ for $|x-y|>t$; c) as $y\to\infty$,

$$\int_{E^n}|\Gamma_t(x,y)|^2\,dx = O\left(\exp\left[c_1 t\sqrt{q^{c_2 t}_{\min}(y)}\right]\right); \quad c_1,\ c_2>0 \text{ are constants;}$$

d) for any $t>0$ the function $\gamma(\lambda,t)$ has a finite number of zeros on every interval $-a\le\lambda\le a$, $|a|<\infty$.

Let us outline the proof of the existence of such a family. Define the sequence of functions

$$\gamma_0(\lambda, t) = 1, \quad \gamma_k(\lambda, t) = \int_0^t \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}\,(t-\tau)\gamma_{k-1}(\lambda, \tau)\, d\tau, \tag{4}$$

$$k \geq 1, \quad 0 \leq t < \infty, \quad -\infty < \lambda < \infty.$$

Conditions a)–d) will be satisfied if, as $\gamma_t(\lambda)$, one takes the functions $\gamma_k(\lambda, t)$ with a sufficiently large index $k = k(n)$.

We dwell on the proof of properties a)–c). We shall use the following assertions.

Lemma 1. Let A be an arbitrary self-adjoint extension in $L_2(E^n)$ of the operator S. Then the domain of definition of the operator

$$\gamma_k(A, t) = \int_{-\infty}^{\infty} \gamma_k(\lambda, t)\, dE_\lambda^A$$

for $k \geq 1$ and $t > 0$ contains all functions from $C_0^\infty(E^n)$. Moreover, for any two extensions A' and A'',

$$\gamma_k(A', t)\varphi = \gamma_k(A'', t)\varphi, \quad \varphi \in C_0^\infty(E^n), \quad t > 0, \quad k \geq 1.$$

Lemma 2. Let A (respectively B) be a self-adjoint extension of the Schrödinger operator with potential $q_1(x)$ (respectively with potential $q_2(x)$). If $q_1(x) = q_2(x)$ in the domain $|x - x_0| \leq R + 2t$, then

$$\gamma_k(A, \tau)\varphi = \gamma_k(B, \tau)\varphi, \quad 0 \leq \tau \leq t, \quad k \geq 1 \tag{5}$$

for all $\varphi \in C_0^\infty(E^n)$ whose support is located in the ball $|x - x_0| \leq R$.

Lemma 3. The operator $\gamma_k(A, t)$, $t > 0$, for $k \geq [(n+1)/2] + 1$, is an integral operator:

$$(\gamma_k(A, t)\varphi)(x) = \int_{E^n} \Gamma_q^{(k)}(t, x, \xi)\varphi(\xi)\, d\xi, \quad \varphi \in C_0^\infty(E^n),$$

and the kernel $\Gamma_q^{(k)}(t, x, \xi)$ vanishes for $|x - \xi| > t$ and is locally bounded jointly in the variables x, ξ.

The proof of Lemmas 1-3 is based on certain facts concerning hyperbolic equations. Namely, let I_q be the operator which assigns to each infinitely differentiable $f(t, x)$ the solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + q(x)u = f(t, x), \qquad u\Big|_{t=0} = \frac{\partial u}{\partial t}\Big|_{t=0} = 0, \tag{6}$$

and let I_q^k be the k-th power of this operator. Using for the solution of problem (6), for example, the method of M. Riesz (^{6}), one can verify that the operators I_q^k are integro-differential, and for sufficiently large k have an integral character. If $\omega_k(x, y; \lambda; t) = \gamma_k(\lambda, t)\psi_A(x, y; \lambda)$, then it is not difficult to show that, for fixed y,

$$\frac{\partial^2 \omega_k}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2 \omega_k}{\partial x_j^2} + q(x)\omega_k = \omega_{k-1}, \qquad \omega_k\Big|_{t=0} = \frac{\partial \omega_k}{\partial t}\Big|_{t=0} = 0, \quad k \geq 1.$$

From these equalities and the uniqueness of the solution of the Cauchy problem (6), it follows that

$$\gamma_k(\lambda, t)\psi_A(x, y; \lambda) = (I_q^k \psi_A(\cdot, y; \lambda))(t, x). \tag{7}$$

Equality (7), in combination with Parseval' s equality, leads to the assertion of Lemma 1; moreover, it turns out that

$$\gamma_k(A, t)\varphi = (I_q^k \varphi)(t, x), \qquad \varphi \in C_0^\infty(E^n). \tag{8}$$

Lemma 2 is a consequence of the fact that the solution $u(t, x)$ of problem (6) at the point (t_0, x_0) depends on the values of $q(x)$ and $f(t, x)$ only in the region $(t_0 - t)^2 - |x_0 - x|^2 \geq 0$, $t_0 \geq t \geq 0$. Lemma 3 follows from equality (8) and the integral character of the operator I_q^k for large k.

From Lemmas 1-3 it follows that, for the family of functions $\gamma_t(\lambda) = \gamma_k(\lambda, t)$, conditions a), b) are satisfied if k is sufficiently large. To prove the estimate

$$\int_{E^n} \left|\Gamma_q^{(k)}(t, x, y)\right|^2 dx = O\left(\exp\left[c_1 t \sqrt{q_{\min}^{ct}(y)}\right]\right), \qquad |y| \to \infty, \tag{9}$$

we first fix a point x_0 and parameters R, t, ε, and apply Lemma 2 in the case where $q_1(x) = q(x)$, $q_2(x) = \chi_{R+2t+\varepsilon}(|x - x_0|)q(x)$; here the function $\chi_{t,\varepsilon}(\lambda) \in C^\infty(E^1)$ is equal to 1 for $0 \leq \lambda \leq t$, to zero for $\lambda \geq t + \varepsilon$, and $0 \leq \chi_{t,\varepsilon}(\lambda) \leq 1$ for $t \leq \lambda \leq t + \varepsilon$. From this choice of $q_2(x)$ it follows that, for the operator B (see equality (5)), the spectrum is located to the right of the point $\lambda = -q_{\min}^{R+2t+\varepsilon}(x_0) = -q_0$. On the basis of Lemma 2 one may assert that

$$\Gamma_q^{(k)}(t, x, y) = \Gamma_{q_2}^{(k)}(t, x, y), \qquad x \in E^n, \quad |y - x_0| \leq R. \tag{10}$$

Let R_z be the resolvent of the operator B; $R_z^{(l)}(x, y)$ the kernel of the l-th power of R_z. Since, with our choice of $q_2(x)$, B is an extension of the Schrödinger operator with bounded potential, for sufficiently large l the relation

$$\int_{E^n} \left| R^{(l)}_{-q_0-1}(x,y) \right|^2 dx = \int_{-q_0}^{\infty} \frac{1}{|\lambda + q_0 + 1|^{2l}} \, \psi_B(y,y;\lambda)\, d\varphi_B(\lambda) \le c, \tag{11}$$

holds, where c is an absolute constant (this follows essentially from the results of A. G. Kostyuchenko (^{4})).

It is not difficult to show that the following equality also holds:

$$\int_{E^n} \left| \Gamma^{(k)}_{q_2}(t,x,y) \right|^2 dx = \int_{-q_0}^{\infty} |\gamma_k(\lambda,t)|^2 \, \psi_B(y,y;\lambda)\, d\rho_B(\lambda). \tag{12}$$

If we take into account that for $l > 0$ one can find $k \ge [(n+1)/2] + 1$ such that

$$|\gamma_k(\lambda,t)|^2 \le M_{kl}(q_0,t)/|\lambda + q_0 + 1|^{2l} \quad \text{for} \quad \lambda \ge -q_0, \tag{13}$$

then from (10), (11), (12), (13) we obtain the inequality

$$\int_{E^n} \left| \Gamma^{(k)}_{q}(t,x,y) \right|^2 dx \le cM\left(q_{\min}^{R+2l+\varepsilon}(x_0), t\right) \tag{14}$$

for $|y - x_0| \le R$.

Estimate (9) follows from the arbitrariness of the point x_0 and the parameters R, ε in (14), and also from the easily verified asymptotic relation $M(q_0,t) = O(e^{ct\sqrt{q_0}})$ as $0 < q_0 \to \infty$ $(c > 0)$.

We have shown that for the functions $\gamma_t(\lambda) = \gamma_k(\lambda,t)$, with sufficiently large k, conditions a)–) are satisfied. The fact that $\gamma_k(\lambda,t)$ also has property) is easily verified using definition (4). Thus, the validity of estimate (3) is proved.

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1. I. E. Shnol, DAN, 94, No. 3, 389 (1954).

2. Yu. M. Berezanskii, Ukr. Math. J., 11, No. 1, 16 (1959).

3. Yu. M. Berezanskii, *Expansion in Eigenfunctions of Self-Adjoint Operators*, Kiev, 1965.
4. A. G. Kostyuchenko, DAN, 115, No. 1, 34 (1957).
5. Yu. M. Berezanskii, Mat. Sbornik, 43, 1, 75 (1957).
6. M. Riesz, Acta Math., 81, No. 1–2 (1949).

Note: Figure translations are in progress. See original paper for figures.