ON SOME LIMITING PROPERTIES OF ORTHOGONAL POLYNOMIALS

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Abstract

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MATHEMATICS

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ON SOME LIMITING PROPERTIES OF ORTHOGONAL POLYNOMIALS

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1. Let the polynomials $\{\varphi_n(z)\}_0^\infty$ be orthonormal on the unit circle

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_n(z)\overline{\varphi_m(z)}\,d\sigma(\theta)=\delta_{nm},\qquad z=e^{i\theta}\quad(m,n=0,1,2,...),\qquad(1)$$

where the set e of points of increase of the bounded nondecreasing function $\sigma(\theta)$ is closed; suppose that on the set $e_1 \Subset e$ there exists $\sigma'(\theta)>0$, and $e_2\subset e_1$ is the set of Lebesgue points of the characteristic function of the set e_1; by $E_2\subset E_1\subset E$ we denote the corresponding sets on the unit circle and by d their transfinite diameters (capacities).

Theorem. *If $d(E_2)=d(E)>0$ and $\mu(\theta)$ is the Robin distribution function for the set E, then at every point $z_0=e^{i\theta_0}$ at which $\mu'(\theta_0)$, $\sigma'(\theta_0)>0$ exist, we have*

$$\lim_{n\to\infty}\frac{K_n(z_0,z_0)}{n+1}=2\pi\,\frac{\mu'(\theta_0)}{\sigma'(\theta_0)},\qquad K_n(z,z_0)=\sum_{s=0}^{n}\varphi_s(z)\overline{\varphi_s(z_0)}\quad(n=0,1,2\,...).\qquad(2)$$

An analogous theorem holds for polynomials orthonormal on a finite interval of the real axis. These results are a generalization of earlier results ([1-7]).

2. We outline the course of the proof of the theorem. Consider the polynomial

$$\Phi_{n+1}(z,z_0)=\frac{K_n(z,z_0)(z-z_0)}{\alpha_n^2\Phi_n(z_0)},\qquad \Phi_n(z)=\frac{\varphi_n(z)}{\alpha_n}=z^n+\dots,\quad(n=0,1,2,...);\qquad(3)$$

all its roots $\{z_k^{(n)}=e^{i\theta_k^{(n)}}\}_0^n$ are simple and lie on $|z|=1$. Let $\psi_n(\theta,\theta_0)$ be a stepwise nondecreasing function having a positive jump $1/(n+1)$ at each point

$\{\theta_k^{(n)}\}_0^n$; consider the infinite sequence of functions $\{\psi_n(\theta, \theta_0)\}_1^\infty$, and suppose that $\theta_0 = \{\theta_0^{(n)}\}_{n=1}^\infty$ is a point of increase of each of them.

We apply the theorem of P. P. Korovkin ([8]) on the existence of the limit

$$\lim_{n\to\infty} M_n^{1/n} = d(E), \qquad M_n = \max|\Phi_n(z)|, \quad z \in E \quad (n = 1, 2, ...); \tag{4}$$

then, slightly modifying Walsh' s theorem ([9], §§ 7.3, 7.4), we prove the existence of the limits

$$\lim_{n\to\infty} |\Phi_n(z)|^{1/n} = \lim_{n\to\infty} |\Phi_{n+1}(z, z_0)|^{1/(n+1)} = d(E)|\varphi(z)|, \qquad |z| > R > 1, \tag{5}$$

where the function $w = \varphi(z)$ maps the region F complementing the set E in the extended plane onto the region $|w| > 1$.

Hence, using Helly' s theorems, we derive convergence at all continuity points of $\mu(\theta)$:

$$\lim_{n\to\infty} \psi_n(\theta, \theta_0) = \mu(\theta). \tag{6}$$

Remark. If one does not use (8), then from our earlier results (5) there follows the validity of (4), and consequently also of (2), under the more restrictive condition: $\sigma'(\theta) > 0$ almost everywhere on the set $E^{(1)}$, where $E^{(1)}$ is the sum of a finite number of arcs of the unit circle, and $E = E^{(1)} + E^{(2)}$, where $E^{(2)}$ is an isolated countable set.

3. Let us now consider a stepwise nondecreasing function $\sigma_n(\theta, \theta_0)$, having $n+1$ points of increase at the same points $\{\theta_k^{(n)}\}_0^n$, and suppose the distributions $d\sigma_n(\theta, \theta_0)$ and $d\sigma(\theta)$ have equal moments $\{c_k\}_0^n$; it is easy to see that the quantity $2\pi\{K_n(z_0, z_0)\}^{-1}$ is equal to the mass of the distribution $d\sigma_n(\theta, \theta_0)$ concentrated at the point $\theta_0^{(n)} = \theta_0$. Let $\theta_0 = \{\theta_0^{(n)}\}_{n=1}^\infty$ be a point of increase of all the functions $\{\sigma_n(\theta, \theta_0)\}_1^\infty$; again using Helly' s theorems and the determinacy of the trigonometric moment problem, we prove convergence in the main,

$$\lim_{n\to\infty} \sigma_n(\theta, \theta_0) = \sigma(\theta). \tag{7}$$

4. Let a small quantity $\varepsilon_n > 0$ be chosen in such a way that both functions $\sigma(\theta)$, $\mu(\theta)$ are continuous at the points $\theta_0 \pm \varepsilon_n$, and that between these points there lies only one point of increase θ_0 of the functions $\sigma_n(\theta, \theta_0)$ and $\psi_n(\theta, \theta_0)$; we have

$$\frac{K_n(z_0, z_0)}{n+1} = 2\pi\,\frac{\psi_n(\theta_0+\varepsilon_n, \theta_0) - \psi_n(\theta_0-\varepsilon_n, \theta_n)}{\sigma_n(\theta_0+\varepsilon_n, \theta_0) - \sigma_n(\theta_0-\varepsilon_n, \theta_0)}; \tag{8}$$

choosing n sufficiently large and using the convergence conditions (6) and (7), we obtain

$$\frac{K_n(z_0, z_0)}{n+1} = 2\pi\,\frac{\mu(\theta_0+\varepsilon_n) - \mu(\theta_0-\varepsilon_n)}{\sigma(\theta_0+\varepsilon_n) - \sigma(\theta_0-\varepsilon_n)} + o(1); \tag{9}$$

dividing the numerator and denominator of the right-hand side by $2\varepsilon_n$ and using the existence of the derivatives, we arrive at (2).

5. Let us consider several examples.

If E is the unit circle, then we have

$$\lim_{n\to\infty} \frac{K_n(z_0, z_0)}{n+1} = \frac{1}{\sigma'(\theta_0)}, \qquad \sigma'(\theta_0) > 0 \tag{10}$$

under the condition $d(E_2) = 1$, or $\sigma'(\theta) > 0$ almost everywhere on the interval $[-\pi, \pi]$; this latter condition is satisfied, in particular, if $\ln\sigma'(\theta) \in L_1(-\pi,\pi)$; but it also holds for Polachek polynomials $((^9), 397\text{–}400)$, for which $\ln\sigma'(\theta) \notin L_1(-\pi,\pi)$.

Now let $E^{(1)}$ be the arc $[\exp\alpha, \exp(2\pi-\alpha)]$, and let the set $E^{(2)}$ be situated on the complementary arc; in this case we have

$$\lim_{n\to\infty} \frac{K_n(z_0, z_0)}{n+1} = \frac{\sin\theta_0/2}{\sigma'(\theta_c)\sqrt{\cos^{(2)}\alpha/2 - \cos^{(2)}\theta/2}}, \qquad \sigma'(\theta_0) > 0, \quad \alpha < \theta_0 < 2\pi - \alpha,$$

under the condition $d(E_2) = \cos\alpha/2$, or $\sigma'(\theta) > 0$ almost everywhere on the interval $[\alpha, 2\pi-\alpha]$.

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Note: Figure translations are in progress. See original paper for figures.