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Abstract

Full Text

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ON THE APPROXIMATION OF STEPANOV ALMOST PERIODIC FUNCTIONS

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1. In this note estimates are given for deviations, in the Stepanov metric, of uniformly almost periodic (u.a.p.) functions with bounded spectra from Stepanov almost periodic (S-a.p.) functions. Some applications of these estimates are indicated to problems concerning the structure and approximation of S-a.p. functions and to the study of convergence of Fourier series; in particular, with their aid A. N. Kolmogorov’ s theorem (^{1,2}) on lacunary Fourier series is generalized to S-a.p. functions.

Let the Fourier series of an S-a.p. function $f(x)$ be written in the following form:

$$f(x) \sim \sum_k A_k e^{i\lambda_k x} \quad (A_k = M\{f(x)e^{-i\lambda_k x}\})\,. \tag{1}$$

Put

$$\|f-g\|_S = \sup_{-\infty<x<\infty} \int_x^{x+1} |f(t)-g(t)|\,dt;$$

$$\omega_S(\delta, f) = \sup_{|h|\le\delta} \left\{ \sup_{-\infty<x<\infty} \int_x^{x+1} |f(t+h)-f(t)|\,dt \right\};$$

by virtue of the S-uniform continuity (^{3}) of $f(x)$,

$$\lim_{\delta\to 0} \omega_S(\delta, f) = 0. \tag{2}$$

It is easy to see that $\omega_S(\delta, f)$ is a nondecreasing subadditive function of δ; therefore, for any $\lambda > 0$,

$$\omega_S(\lambda\delta, f) \le (1+\lambda)\omega_S(\delta, f). \tag{3}$$

Introduce the function

$$\varphi_{\lambda,\mu}(t) = \begin{cases} 1, & |t| \le \lambda, \\ \dfrac{1}{2} - \dfrac{2}{\mu-\lambda}\left(|t| - \dfrac{\lambda+\mu}{2}\right) - \dfrac{2}{(\mu-\lambda)^2}\left(|t| - \dfrac{\lambda+\mu}{2}\right)^2, & \lambda < |t| \le \dfrac{\lambda+\mu}{2}, \\ \dfrac{1}{2} - \dfrac{2}{\mu-\lambda}\left(|t| - \dfrac{\lambda+\mu}{2}\right) + \dfrac{2}{(\mu-\lambda)^2}\left(|t| - \dfrac{\lambda+\mu}{2}\right)^2, & \dfrac{\lambda+\mu}{2} < |t| \le \mu, \\ 0, & |t| > \mu, \end{cases}$$

and its Fourier transform

$$\Psi_{\lambda,\mu}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{\lambda,\mu}(t) e^{-iut}\, dt = \frac{16}{\pi(\mu-\lambda)^2} \frac{\sin^2 \dfrac{\mu-\lambda}{4} u \, \sin \dfrac{\mu+\lambda}{2} u}{u^3}.$$

The estimates hold

$$\int_{-\infty}^{\infty} |\Psi_{\lambda,\mu}(u)|\, du \le A \ln \frac{\mu+\lambda}{\mu-\lambda} + B, \qquad \int_0^{\infty} u |\Psi_{\lambda,\mu}(u)|\, du \le \frac{4A}{\mu-\lambda}; \tag{4}$$

where $A = \dfrac{2}{\pi}$, $B = \dfrac{3}{\pi} + \dfrac{2}{\pi}\ln 2$; their proof is analogous to the proof of Lemma 1.10.1 from (^{3}).

As a consequence of the Fourier inversion formula,

$$\varphi_{\lambda,\mu}(\Lambda) = \int_{-\infty}^{\infty} \Psi_{\lambda,\mu}(u) e^{-i\Lambda u}\, du$$

and, in particular,

$$\int_{-\infty}^{\infty} \Psi_{\lambda,\mu}(u)\, du = \varphi_{\lambda,\mu}(0) = 1. \tag{5}$$

2. Let us formulate the principal theorem of the note.

Theorem 1. If $f(x)$ is an S-a.p. function having the Fourier series (1), then

$$f_{\lambda,\mu}(x) = \int_{-\infty}^{\infty} f(x+u)\Psi_{\lambda,\mu}(u)\, du$$

is a uniformly a.p. function with Fourier series

$$f_{\lambda,\mu}(x) \sim \sum_k A_k \varphi_{\lambda,\mu}(\lambda_k) e^{i\lambda_k x} \tag{6}$$

and the estimate

$$\|f_{\lambda,\mu} - f\|_S \leq \omega_S\left(\frac{1}{\mu-\lambda}, f\right)\left[A \ln \frac{\mu+\lambda}{\mu-\lambda} + C\right], \tag{7}$$

where $C = B + 8A$.

The proof of the first assertion of the theorem is constructed according to the scheme of the proof of Lemma 1.10.2 from (^{3}); (7) follows from (3), (4), (5), and known (^{3}) properties of S-a.p. functions.

We note that, by the definition of $\varphi_{\lambda,\mu}(t)$, the spectrum of the function $f_{\lambda,\mu}(x)$ belongs to the interval $(-\mu, \mu)$, and, by (2), $\|f_{\lambda,\mu} - f\|_S \to 0$ as $\mu \to \infty$, if $\lambda/\mu \leq \theta < 1$.

3. Let

$$E_\mu^{(S)}(f) = \inf_{g \in B_\mu} \{\|f - g\|_S\},$$

where B_μ is the class of entire functions of degree $\leq \mu$, bounded on the real axis.

Theorem 2. For an S-a.p. function $f(z)$,

$$E_\mu^{(S)}(f) \leq C\omega_S\left(\frac{1}{\mu}, f\right). \tag{8}$$

Theorem 3. For an S-a.p. function $f(x)$,

$$\|f_{\lambda,\mu} - f\|_S \leq E_\lambda^{(S)}(f)\left[A \ln \frac{\mu+\lambda}{\mu-\lambda} + B + 1\right]. \tag{9}$$

Theorem 4. If the Fourier exponents of the S-a.p. function $f(x)$ with Fourier series (1) have a unique limit point at infinity, then

$$\|f - S_\lambda\|_S \leq E_\lambda^{(S)}(f)\left[A \ln \frac{\mu+\lambda}{\mu-\lambda} + N_f(\mu) - N_f(\lambda) + B + 1\right], \tag{10}$$

where

$$S_\lambda(x) = \sum_{|\lambda_k| \leq \lambda} A_k e^{i\lambda_k x}, \qquad N_f(\lambda) = \sum_{|\lambda_k| \leq \lambda} 1.$$

Concerning the proof of these theorems we note the following. Estimate (8) follows from (7), since $f_{0,\mu}(x) \in B_\mu$. The proof of estimate (9) is based to a considerable extent on (4) and on the application of the equality $g_{\lambda,\mu}(x) = g(x)$, which holds (^{4,6}) for any function $g(x) \in B_\lambda$. On the basis of Lemma 2 of paper (^{6}), $|A_k| \le E_\lambda^{(S)}(f)$ if $|\lambda_k| > \lambda$; therefore, when fulfilled

conditions of Theorem 4

$$\sup_x |f_{\lambda,\mu}(x) - S_\lambda(x)| \le E_\lambda^{(S)}(f)[N_f(\mu) - N_f(\lambda)], \tag{11}$$

and (10) follows from (9) and (11).

Theorem 4 is an analogue of Theorem 1 of the paper [7], which gives an estimate of the deviation of a partial sum of the Fourier series from a uniformly almost periodic function in the space $C(-\infty, \infty)$.

In view of the topological equivalence of the S-distances

$$\|f-g\|_S^{(\alpha)} = \sup_{-\infty<x<\infty} \left\{ \frac{1}{\alpha} \int_x^{x+\alpha} |f(t)-g(t)|\,dt \right\} \quad (\|f-g\|_S = \|f-g\|_S^{(1)})$$

the estimate (10), after the corresponding specification, becomes applicable for establishing convergence criteria for Fourier series of 2π-periodic functions in the space $\mathcal{L}(0, 2\pi)$ [2].

4. Theorem 5. Let x be a Lebesgue point of the S-a.p. function $f(x)$; then

$$f_{\mu/2,\mu}(x) - f(x) = o(1) \qquad (\mu \to \infty). \tag{12}$$

We outline the proof of this theorem. By the properties of the kernel $\Psi_{\lambda,\mu}(u)$ and the S-boundedness (3) of the function $f(x)$,

$$f_{\mu/2,\mu}(x) - f(x) = \int_0^1 \varphi_x(u)\Psi_{\mu/2,\mu}(u)\,du + O\left(\frac{1}{\mu^2}\right), \tag{13}$$

where $\varphi_x(u) = f(x+u) + f(x-u) - 2f(x)$.

At every Lebesgue point $\Phi_x(u) = \int_0^u |\varphi_x(t)|\,dt = o(u)$, therefore

$$\int_0^{1/\mu} |\varphi_x(u)|\,|\Psi_{\mu/2,\mu}(u)|\,du \le \frac{3}{4}\mu\Phi_x\left(\frac{1}{\mu}\right) = o(1). \tag{14}$$

Integrating by parts the right-hand side of the inequality

$$\int_{1/\mu}^{1} |\varphi_x(u)|\,|\Psi_{\mu/2,\mu}(u)|\,du \leq \frac{64}{\pi\mu^2}\int_{1/\mu}^{1} |\varphi_x(u)|\,\frac{du}{u^3},$$

we obtain the estimate

$$\int_{1/\mu}^{1} |\varphi_x(u)|\,|\Psi_{\mu/2,\mu}(u)|\,du = o(1). \tag{15}$$

From (13), (14), and (15) follows (12).

Theorem 6. If the Fourier exponents of the S-a.p. function $f(x)$ with Fourier series (1) have a unique limit point at infinity, x is a Lebesgue point of the function $f(x)$, and

$$E_{\mu}^{(S)}(f)[N_f(2\mu) - N_f(\mu)] = o(1) \qquad (\mu \to \infty), \tag{16}$$

then

$$S_\mu(x) - f(x) = o(1) \qquad (\mu \to \infty). \tag{17}$$

Proof. By (11) and (16), at any point x,

$$|f_{\mu/2,\mu}(x) - S_{\mu/2}(x)| \leq E_{\mu/2}^{(S)}(f)[N_f(\mu) - N_f(\mu/2)] = o(1). \tag{18}$$

From (12) and (18) we obtain (17).

An increasing sequence of positive numbers $L = \{l_k\}$ $(k = 1, 2, ...)$ will be called θ, s-lacunary if there exist a natural number s and $\theta > 1$ such that $l_{k+s}/l_k \geq \theta$ $(k = 1, 2, ...)$.

Denote by $L(f)$ the set of absolute values of the Fourier exponents of the function $f(x)$. We shall call the series (1) θ, s-lacunary if it contains no constant term, the set $L(f)$ has a unique limit point at infinity and, when renumbered in increasing order of its elements, forms a θ, s-lacunary sequence.

The condition

$$N_f(2\mu) - N_f(\mu) = O(1) \tag{19}$$

is necessary and sufficient (^{5,7}) for the series (1) to be θ, s-lacunary.

By virtue of (19), (8), (2), and (18), the following generalization of A. N. Kolmogorov' s theorem holds:

Theorem 7. *If the Fourier series of an S-a.p. function $f(x)$ is θ, s-lacunary, then it converges to $f(x)$ almost everywhere.*

5. From Theorem 5 and the uniqueness theorem for uniformly a.p. functions (^{3}) there follows the uniqueness theorem for S-a.p. functions (for S_2-a.p. functions it is true by Parseval' s equality (^{3})).

Theorem 8. *If all Fourier coefficients of an S-a.p. function $f(x)$ are equal to zero, then $f(x) = 0$ almost everywhere.*

We note that this assertion can be obtained as an immediate consequence of Theorem 1.

Below we give two theorems which make it possible to establish that certain S-a.p. functions belong to the class of uniformly a.p. functions (here equivalent functions are regarded as identical).

Theorem 9. *An S-a.p. function with bounded spectrum is a uniformly a.p. function.*

This theorem follows from Theorems 1 and 8.

Theorem 10. *A bounded S-a.p. function $f(x)$ is a uniformly a.p. function if one of the following conditions is satisfied: 1) the Fourier exponents of $f(x)$ are linearly independent; 2) the Fourier coefficients of $f(x)$ are nonnegative.*

Proof. It is enough, in view of Theorem 8, to prove the absolute convergence of the Fourier series (1) of the function $f(x)$; under each of the two conditions of the theorem this follows from the estimate

$$\sum_{|\lambda_k| \le \mu/2} |A_k| \le \sup_x |f_{\mu/2,\mu}(x)| \le (A \ln 3 + B) \sup_x |f(x)|,$$

which follows from (4), (6), and the known (^{3}) criteria for absolute convergence of Fourier series of uniformly a.p. functions.

Theorem 10 can be strengthened if the set $L(f)$ is not everywhere dense. Indeed, if the interval $(\sigma, \sigma+\varepsilon)$ $(\sigma > 0,\ \varepsilon > 0)$ contains no points of the set $L(f)$, then, by Theorems 1 and 9, it is enough that either of the two conditions of Theorem 10 be satisfied for the S-a.p. function

$$\tilde{f}_\sigma(x) \sim \sum_{|\lambda_k| > \sigma} A_k e^{i\lambda_k x}.$$

In conclusion we note that, as a consequence of Theorem 1 of the paper (^{8}) and Theorem 8, a bounded S-a.p. function with a θ, s-lacunary Fourier series turns out to be a uniformly a.p. function.

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Note: Figure translations are in progress. See original paper for figures.