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Abstract

Full Text

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AN UPPER ESTIMATE OF THE ENTROPY OF A CLASSICAL DYNAMICAL SYSTEM

(Presented by Academician A. N. Kolmogorov, October 7, 1964)

A classical dynamical system will mean a measure-preserving diffeomorphism T of an n -dimensional compact Riemannian manifold M (the measure μ is defined by the Riemannian metric, and $\mu(M) = 1$).

Let Δ be an $(n - 1)$ -dimensional element on M . Its $(n - 1)$ -dimensional volume (unoriented) will henceforth be called its area and denoted by $S(\Delta)$. The transformation T changes $S(\Delta)$. From the compactness of M it follows that the coefficient of expansion of area is bounded above by some number. Denote this number by λ , so that

$$S(T\Delta) \leq \lambda S(\Delta). \tag{1}$$

(In what follows we shall adhere to the notation of the survey ⁽¹⁾.)

Theorem 1. *The entropy $h(T)$ of a classical dynamical system is finite and*

$$h(T) \leq n \log \lambda \tag{2}$$

(the logarithm is binary).

Outline of the proof. We shall call ξ a **smooth partition** if ξ is a partition of M into a finite number of sets with piecewise smooth boundaries, and the sum of the areas of the boundaries of these sets is finite. We shall call the sum of these areas the **area** of the partition ξ and denote it by $S(\xi)$. (With this definition, each piece of boundary is counted twice, since it is the boundary of two elements of the partition.)

The entropy $h(T, \xi)$ of a smooth partition ξ is $\lim_{N \rightarrow \infty} H(\eta_N)/N$, where $\eta_N = \xi \cdot T\xi \cdot \dots \cdot T^{N-1}\xi$. With the aid of a certain analogue of the isoperimetric inequality it will be shown that $H(\eta_N) \leq n \log S(\eta_N) + C_1$, while from (1) it follows that $\log S(\eta_N) < N \log \lambda + C_2$, where C_1, C_2 do not depend on N . Hence $h(T, \xi) \leq n \log \lambda$ for any smooth ξ ; and since any partition can be approximated by smooth ones, it follows that $h(T) \leq n \log \lambda$.

For convenience, we divide the proof itself into a number of points.

1. Lemma. Suppose

$$\sum_{i=1}^k \mu_i = 1, \quad (3a)$$

$$\mu_i > 0; \quad i = 1, \dots, k. \quad (3b)$$

Then

$$-\sum_{i=1}^k \mu_i \log \mu_i \leq n \log \sum_{i=1}^k \mu_i^{(n-1)/n} \quad (n \geq 2). \quad (4)$$

For the proof, fix $\bar{\mu}$ and consider, as a function of $\vec{v} = (v_1, v_2, \dots, v_k)$,

$$F(\vec{v}) = \sum_{i=1}^k (\mu_i \log \mu_i - \mu_i \log v_i),$$

where \vec{v} satisfies the conditions $\sum_{i=1}^k v_i = 1$, $v_i > 0$. It is easy to verify that the minimum of $F(\vec{v})$ is attained at the point $v_i = \mu_i$, i.e.

$$\sum_{i=1}^k (\mu_i \log \mu_i - \mu_i \log v_i) \geq 0.$$

Putting now

$$v_i = \mu_i^{(n-1)/n} / \sum_{i=1}^k \mu_i^{(n-1)/n},$$

we obtain (4).

2. The following assertion is proved without difficulty. There exists a smooth partition ξ_0 of the manifold M such that for every set with piecewise smooth boundary, lying entirely in one of the elements of ξ_0 , the isoperimetric inequality

$$V^{(n-1)/n} \leq CS, \quad (5)$$

holds, where V is the volume of the set, S is the area of its boundary, and C depends on the original manifold M and on the choice of ξ_0 , but does not depend on the set.

3. In [1] it is proved that if a sequence of finite measurable partitions of the space M has the following two properties:

$$\xi_0 \leq \xi_1 \leq \xi_2 \leq \dots, \quad (6)$$

$$\prod_{i=0}^{\infty} \xi_i = 0 \pmod{0}, \quad (7)$$

then for any measure-preserving transformation T of the space M ,

$$h(T) = \lim_{i \rightarrow \infty} h(T, \xi_i). \quad (8)$$

4. For the given M and the ξ_0 already constructed in item 2, it is easy to construct a sequence of smooth partitions ξ_1, ξ_2, \dots which will satisfy conditions (6), (7). Let ξ be one of the ξ_i . From (1) it is clear that $S(T\xi) \leq \lambda S(\xi)$, $S(T^2\xi) \leq \lambda^2 S(\xi)$, \dots , $S(T^{N-1}\xi) \leq \lambda^{N-1} S(\xi)$. For

$$\eta_N = \xi \cdot T\xi \cdot \dots \cdot T^{N-1}\xi$$

we obtain

$$S(\eta_N) \leq (1 + \lambda + \dots + \lambda^{N-1})S(\xi) < \frac{S(\xi)}{\lambda - 1} \lambda^N. \quad (9)$$

Now note that each element of the partition η_N lies in one of the elements of the partition ξ_0 , and therefore the area and volume of this element are related by inequality (5). Denoting now the volume of the i -th element of the partition η_N by μ_i , in view of (5), (9), we shall have

$$\sum \mu_i = 1, \quad \mu_i > 0, \quad \sum \mu_i^{(n-1)/n} \leq CS(\eta_N) < \frac{CS(\xi)}{\lambda - 1} \lambda^N.$$

Hence, by the lemma,

$$H(\eta_N) \leq n \log \left(\frac{CS(\xi)}{\lambda - 1} \lambda^N \right), \quad \lim_{N \rightarrow \infty} \frac{H(\eta_N)}{N} \leq n \log \lambda,$$

i.e. $h(T, \xi_i) \leq n \log \lambda$.

Now from (8) it follows that $h(T) \leq n \log \lambda$. Theorem 1 is proved.

Theorem 1 is also true for endomorphisms (the quantity λ is determined by the condition $S(T^{-1}\Delta) \leq \lambda S(\Delta)$, where $T^{-1}\Delta$ is the full preimage of Δ). In the case of an endomorphism the estimate (2) is attained in every dimension.

Theorem 1 is also true for manifolds with boundary having finite area, if estimate (1) is valid for T . In this class of dynamical systems, in every dimension there exist systems for which $h(T) = (n - 1) \log \lambda$.

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REFERENCES

1. V. A. Rokhlin, *UMN*, 15, no. 4 (94), 3 (1960).

Note: Figure translations are in progress. See original paper for figures.

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