ON THE ELEMENTARY THEORY OF MAXIMAL NORMED FIELDS

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Abstract

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MATHEMATICS

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ON THE ELEMENTARY THEORY OF MAXIMAL NORMED FIELDS

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1. We shall consider objects of the form $\langle F, v, \Gamma\rangle$, where F is a field; v is a (non-Archimedean) norming of the field F; Γ is the value group of the norming v. We shall call these objects, not quite precisely, normings. F_v will denote the residue field of the norming v. The notation $\langle F_1, v_1, \Gamma_1\rangle \subseteq \langle F_2, v_2, \Gamma_2\rangle$ will mean that $F_1 \subseteq F_2$, $\Gamma_1 \subseteq \Gamma_2$, $v_1(a) = v_2(a)$ for $a \in F_1$; in this case the norming v_2 is called an extension of the norming v_1. If $n = [F_2 : F_1]$ is finite, then $e = [\Gamma_2 : \Gamma_1]$ is the ramification index of the norming and $f = [(F_2)_{v_2} : F_{1v_1}]$ is the relative degree of the norming, finite, and $n \geq f \cdot e$ (^{2}).

If $\langle F_1, v_1, \Gamma_1\rangle \subseteq \langle F_2, v_2, \Gamma_2\rangle$ and $F_{2v_2} = F_{1v_1}$, $\Gamma_2 = \Gamma_1$, then v_2 is called an immediate extension of v_1. If a norming has no proper immediate extensions, then it is called maximal. Examples of maximal normings are provided by fields of formal power series $\langle F(t^\Gamma), v, \Gamma\rangle$ (^{3}) and local fields (^{4}).

We shall call a norming $\langle F, v, \Gamma\rangle$ algebraically complete if: 1) for every finite algebraic extension F_1 of the field F there exists a unique (up to the naturally defined equivalence of normings over $\langle F, v, \Gamma\rangle$) norming of the field F_1 such that $\langle F, v, \Gamma\rangle \subseteq \langle F_1, v_1, \Gamma_1\rangle$; 2) for every finite algebraic extension F_1 of the field F and norming $\langle F_1, v_1, \Gamma_1\rangle$ satisfying condition 1), the equality $n = f \cdot e$ holds, where $n = [F_1 : F]$, e is the ramification index, f is the relative degree.

Lemma. *If $\langle F, v, \Gamma\rangle$ is a maximal norming, then it is algebraically complete.*

Let $\langle F, v, \Gamma\rangle$ be a norming; a well-ordered sequence $\{a_\rho\}$ of elements of F, having no last element, is called pseudo-convergent if $v(a_\rho - a_\sigma) > v(a_\sigma - a_\tau)$ for $\rho > \sigma > \tau$. An element $a \in F$ is called a limit of the pseudo-convergent sequence $\{a_\rho\}$ if $v(a - a_\rho) = v(a_{\rho+1} - a_\rho)$ (^{3}). A norming $\langle F, v, \Gamma\rangle$ is called ξ-complete, where ξ is an ordinal number, if every pseudo-convergent sequence $\{a_\rho\}_{\rho<\beta\leq\xi}$ of elements of F has a limit in F (^{3}). If ξ is the least ordinal of cardinality $\aleph_\alpha$, then instead of ξ-completeness we shall speak of $\aleph_\alpha$-completeness.

2. We shall regard normings $\langle F, v, \Gamma\rangle$ as models of the signature

$$\sigma = \langle F^1, \Gamma^1, V^2, Q^2, {S'}^3, S^3, P^3 \rangle,$$

where the predicate F^1 singles out the elements of F, Γ^1 the elements of Γ; $V^2(x, y)$ means that $x \in F$, $y \in \Gamma$, and $v(x) = y$; Q^2 is the order relation $\leq$ on Γ; ${S'}^3$ is the addition predicate in Γ; S^3, P^3 are the predicates of addition and multiplication in the field F.

Let T_0 be a system of first-order sentences of the signature σ, whose models are precisely all algebraically complete normings. T_0 can be chosen recursively (^{1}).

If $\mathfrak{A}$ is an arbitrary formula, then $\mathfrak{A}^\Gamma$ will denote the formula obtained from $\mathfrak{A}$ by relativizing the quantifiers with respect to the predicate Γ^1. If T is a set of formulas, then $T^\Gamma = \{\mathfrak{A}^\Gamma \mid \mathfrak{A} \in T\}$. If $\mathfrak{A}$ is arbitrary—

free formula, then $\mathfrak{A}^{F_v}$ will denote the formula obtained from $\mathfrak{A}$ by relativizing the quantifiers with respect to the formula

$$A_v(x) = V^2(x, y) \ \&\ Q^2(0, y) \ (v(x) \geq 0)$$

and replacing $x = y$ by

$$(\exists z) S^3(y, z, x) \ \&\ (\exists t)(V^2(z, t) \ \&\ Q^2(0, t) \ \&\ (t \neq 0)) \quad (v(x - y) > 0).$$

If T is a set of formulas, then

$$T^{F_v} = \{\mathfrak{A}^{F_v} \mid \mathfrak{A} \in T\}.$$

3. **Theorem 1.** Let $\langle F, v, \Gamma \rangle$ and $\langle F_1, v_1, \Gamma_1 \rangle$ be two algebraically complete normings, $F_v \simeq F_{1v_1}$ fields of characteristic 0, $\Gamma \simeq \Gamma_1$, $\mathrm{Ext}^1(A, F_v^*) = 0$ for every torsion-free abelian group A. If $\overline{F} = \overline{F_1} = \aleph_{\alpha+1}$ and $\langle F, v, \Gamma \rangle, \langle F_1, v_1, \Gamma_1 \rangle$ are $\aleph_\alpha$-complete, then $\langle F, v, \Gamma \rangle$ is isomorphic to $\langle F_1, v_1, \Gamma_1 \rangle$.

Theorem 2. Let $\langle F, v, \Gamma \rangle$ and $\langle F_1, v_1, \Gamma_1 \rangle$ be two algebraically complete normings; F_v a field of characteristic 0. $\langle F, v, \Gamma \rangle$ is arithmetically equivalent to $\langle F_1, v_1, \Gamma_1 \rangle$ if and only if F_v is arithmetically equivalent to F_{1v_1}, and Γ is arithmetically equivalent to Γ_1.

The proof of Theorems 1 and 2 uses the results of the papers (^{1,3,4}), and the idea of the proof of these theorems is analogous to the idea of the proof of the corresponding theorems in (^{3}); in particular, the proof uses the technique of ultraproducts.

Theorem 3. Let $\mathfrak{F}$ be a class of fields of characteristic 0; $\mathfrak{G}$ a class of ordered abelian groups;

$$\mathfrak{F}(\mathfrak{G}) = \{\langle F, v, \Gamma\rangle \mid \langle F, v, \Gamma\rangle$$

is a maximal norming, $F_v \in \mathfrak{F}$, $\Gamma \in \mathfrak{G}\}$. If T_1 is a system of axioms of the theory of the class $\mathfrak{F}$; T_2 is a system of axioms of the theory of the class $\mathfrak{G}$, then

$$T_0 \cup T_1^{F_v} \cup T_2^{\Gamma}$$

is a system of axioms of the class $\mathfrak{F}(\mathfrak{G})$.

Theorem 4. Let $\mathfrak{F}$ be a class of fields of characteristic 0, $\mathfrak{G}$ a class of ordered abelian groups. If the theory of the class $\mathfrak{F}$ is decidable, and the theory of the class $\mathfrak{G}$ is decidable, then the theory of the class $\mathfrak{F}(\mathfrak{G})$ is also decidable.

4. Let F be a field of characteristic $p \neq 0$. We shall say that F **satisfies assumption A** (^{4}) if, for every equation of the form

$$x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{nx} = b, \qquad a_i,\ b \in F,$$

there is at least one root in F.

Theorem 1′. Let $\langle F, v, \Gamma\rangle$ and $\langle F_1, v_1, \Gamma_1\rangle$ be two algebraically complete normings; $F_v \simeq F_{1v_1}$; F, F_1 fields of characteristic $p \neq 0$; F_v satisfies assumption A; $\Gamma \simeq \Gamma_1$; $p\Gamma = \Gamma$; $\mathrm{Ext}^1(A, F_v^*) = 0$ for every torsion-free abelian group A. If $\overline{F_1} = \overline{F} = \aleph_{\alpha+1}$ and $\langle F, v, \Gamma\rangle, \langle F_1, v_1, \Gamma_1\rangle$ are $\aleph_\alpha$-complete, then $\langle F, v, \Gamma\rangle$ is isomorphic to $\langle F_1, v_1, \Gamma_1\rangle$.

Theorem 2′. Let $\langle F, v, \Gamma\rangle$ and $\langle F_1, v_1, \Gamma_1\rangle$ be two algebraically complete normings; F, F_1, F_v fields of characteristic $p \neq 0$; F_v satisfies assumption A; $p\Gamma = \Gamma$. $\langle F, v, \Gamma\rangle$ is arithmetically equivalent to $\langle F_1, v_1, \Gamma_1\rangle$ if and only if F_v is arithmetically equivalent to F_{1v_1}, and Γ is arithmetically equivalent to Γ_1.

Theorem 3′. Let $\mathfrak{F}$ be a class of fields of characteristic $p \neq 0$ satisfying assumption A; $\mathfrak{G}$ a class of ordered abelian groups Γ such that $p\Gamma = \Gamma$;

$$\mathfrak{F}(\mathfrak{G}) = \{\langle F, v, \Gamma\rangle \mid \langle F, v, \Gamma\rangle$$

is a maximal norming, F is a field of characteristic p, $F_v \in \mathfrak{F}$, $\Gamma \in \mathfrak{G}\}$. If T_1 is a system of axioms of the theory of the class $\mathfrak{F}$; T_2 is a system of axioms of the class $\mathfrak{G}$, then

$$T_0 \cup T_1^{F_v} \cup T_2^{\Gamma} \cup \{(\forall x)(px = 0)\}$$

is a system of axioms of the theory of the class $\mathfrak{F}(\mathfrak{G})$.

Theorem 4′. Let $\mathfrak{F}$ be a class of fields of characteristic $p \neq 0$ satisfying assumption A; $\mathfrak{G}$ a class of ordered abelian groups Γ such that $p\Gamma = \Gamma$. If the theory of the class $\mathfrak{F}$ is decidable, and the theory of the class $\mathfrak{G}$ is decidable, then the theory of the class $\mathfrak{F}(\mathfrak{G})$ is decidable.

5. All ordered abelian groups Γ considered in this section will satisfy the following restriction:

There exists a least positive element in Γ.

$$(*)$$

We shall denote this element by l.

We shall say that the norming $\langle F, v, \Gamma\rangle$ is **absolutely unramified** if, in the case when F_v has characteristic $p \neq 0$,

$$v(p) = v(\underbrace{1 + \cdots + 1}_{p \text{ times}}) = I.$$

Theorem 2″. Let $\langle F, v, \Gamma\rangle$ and $\langle F_1, v_1, \Gamma_1\rangle$ be two algebraically complete absolutely unramified normings. $\langle F, v, \Gamma\rangle$ is arithmetically equivalent to $\langle F_1, v_1, \Gamma_1\rangle$ if and only if F_v is arithmetically equivalent to F_{1v_1}, and Γ is arithmetically equivalent to Γ_1.

Let $T_0' = \{\mathfrak{A}_p \mid p$ is a prime number; $\mathfrak{A}_p$ is the formula meaning that if $v(p) > 0$, then $v(p) = I\}$.

Theorem 3″. Let $\mathfrak{F}$ be a class of fields; $\mathfrak{G}$ a class of ordered abelian groups satisfying condition $(*)$; $\mathfrak{F}(\mathfrak{G}) = \{\langle F, v, \Gamma\rangle \mid \langle F, v, \Gamma\rangle$ is an absolutely unramified maximal norming; $F \in \mathfrak{F},\ \Gamma \in \mathfrak{G}\}$. If T_1 is a system of axioms for the theory of the class $\mathfrak{F}$; T_2 is a system of axioms for the theory of the class $\mathfrak{G}$, then $T_0 \cup T_0' \cup T_1^F \cup T_2^\Gamma$ is a system of axioms for the theory of the class $\mathfrak{F}(\mathfrak{G})$.

Theorem 4″. Let $\mathfrak{F}$ be a class of fields, $\mathfrak{G}$ a class of ordered abelian groups satisfying condition $(*)$. If the theory of the class $\mathfrak{F}$ is decidable and the theory of the class $\mathfrak{G}$ is decidable, then the theory of the class $\mathfrak{F}(\mathfrak{G})$ is decidable.

6. All results on the decidability of elementary theories of fields obtained in ([1],[3]) are consequences of the theorems indicated above. For example, from Theorem 4″ it follows that, for any prime number p, the theory of the field Q_p of p-adic numbers is decidable.

Theorems 4, 4′, 4″ make it possible to indicate many different fields and classes of fields with decidable theory; for example, an infinite field of characteristic $p \neq 0$ with decidable theory that is not algebraically closed. The class of fields of formal power series $\{F, F\{t_1\}, \dots, F\{t_1\}, \dots, \dots \{t_n\}, \dots\}$ has a decidable theory if the field F of characteristic 0 has a decidable theory, or the class of fields of formal power series $\{F(t^\Gamma) \mid \Gamma \in \mathfrak{G}\}$, if F is a field of characteristic 0 with decidable theory, $\mathfrak{G}$ is an arbitrary class of ordered abelian groups with decidable theory, etc.

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Note: Figure translations are in progress. See original paper for figures.