V. P. PLATONOV

1965

SovietRxiv

Abstract

Full Text

V. P. PLATONOV

ON THE THEORY OF TOPOLOGICAL GROUPS

(Presented by Academician A. I. Mal' cev on 21 XII 1964)

§ 1.

Following ([1]), we shall call a topological group G **locally projectively nilpotent**, or **locally nilpotent in the topological sense**, if every finitely generated subgroup of it is projectively nilpotent in the induced topology. It was observed by the author in ([1]) and independently by V. I. Ushakov that the main results of V. M. Glushkov ([2]) on the structure of locally nilpotent locally bicompact groups carry over to the more general case of locally projectively nilpotent groups. However, the analogy between the abstract and the topological cases cannot be regarded as complete until the following natural question is resolved: can the term "finite" in the topological case be replaced by "bicompact" without detriment to the theory? For example, can finite generation in the definition of locally projectively nilpotent groups be replaced, without loss of generality, by bicompact generation?

The latter question is specific to locally projectively nilpotent groups and requires the introduction of new ideas as compared with ([2]), whereas the direct generalization of the results from ([2]) is achieved by the same means. In essence, the answer to this question is equivalent to proving a certain independent theorem on projective solvability, which no longer extends to solvable groups and even to topological N-groups. In what follows only locally bicompact groups are considered, which will not be stipulated separately.

The answer to the question formulated above is given by

Theorem 1. *A bicompactly generated locally projectively nilpotent group G is projectively nilpotent.*

Proof. We shall show that the group G may be assumed to satisfy the following conditions: a) the group G satisfies the second axiom of countability; b) the subgroup B of all bicompact elements of G is nilpotent; c) the connected component of the identity B_0 is a commutative Lie group.

Condition a) may be assumed to hold in view of the Gleason–Kakutani theorem (see ([3,4])).

According to Theorem 9 of ([1]), the set B of bicompact elements forms a closed subgroup of G, which is bicompact by Theorem 1 of ([5]), while B_0 is central in G_0 and B. Suppose condition a) is fulfilled for the group G. Then B_0 is a connected bicompact group with the second axiom of countability. From ([6]), § 38, it follows that in this case B_0 has a dense subgroup with two generators. At the same time, by Theorem 9 of ([1]), B_0 is central in $N = B \cdot G_0$, and $B \subset Z_G(G_0)$. Since the subgroup N is open, the quotient group $G^* = G/N$ is a discrete finitely generated nilpotent group. If $g_1, g_2, \dots, g_k$ are representatives of the classes generating the group G^*, then the group $R = \{g_1, \dots, g_k\}$ is projectively nilpotent by assumption. Hence, from Theorem 11.7 of ([2]) it follows easily that R is projectively solvable, i.e. in $U \ni e$ there is such a normal divisor H_u that R/H_u is a Lie group. Similarly, the group $\widetilde{R} = \{R, B_0\}$ is also projectively solvable and, in view of the arbitrariness of U, B_0 may be assumed to be a Lie group, i.e. condition b) is fulfilled.

Finally, condition c) is easily achieved if one takes into account that the group B is projectively nilpotent.

Thus, let the group G satisfy conditions a), b), c). Since B_0 is a Lie group, G_0 is also a Lie group, and it is not hard to see that in G_0 there is a dense finitely generated subgroup $F = \{f_1, \dots, f_r\}$. Then

$$G = \overline{\{g_1, \dots, g_k, f_1, \dots, f_r, B\}}.$$

Consider the groups

$$H_{g,f,b} = \{g_1, \dots, g_k, f_1, \dots, f_r, b_1, \dots, b_m\}$$

for all possible finite numbers of $b_i \in B$. Obviously, $H_{g,f,b}$ are projectively nilpotent and

$$\bigcup_{b \in B} H_{g,f,b} = G.$$

By Theorem 11.7 from (2), the group $H_{g,f,b}$ is projectively Lie; consequently, there is a normal divisor $L_{ub} \subset U_b$ such that $H_{g,f,b}/L_{ub}$ is a Lie group, where we assume that all U_b are induced by the neighborhood U. Then all $L_{ub} \subset U$. The group $N = B \cdot G_0$ is open in G, and therefore one may assume that $U = U_B \cdot U_{G_0}$. In turn, since B is finite-dimensional, B has a full system of neighborhoods of the identity decomposable into a direct product of a zero-dimensional normal divisor and a local Lie group. Hence it follows that $U_B = F_u \cdot U_{B_0}$, where F_u is a zero-dimensional normal divisor of B. $U = F_u \cdot U_{B_0} \cdot U_{G_0}$; choose U_{B_0} and U_{G_0} so that in $U_{B_0} \cdot U_{G_0}$ there are no nontrivial subgroups. Then F_u is the only subgroup in U, for if $g = f \cdot u$, where $f \in F_u$, $u \in U_{B_0} \cdot U_{G_0}$, then for some n, $u^n \in U_{B_0} \cdot U_{G_0}$; hence

$$g^n = f^n \cdot u^n = f_1 \cdot u_1,$$

whence

$$u^n = f^{-n} f_1 u_1 = u_1.$$

Thus, if $U = F_u \cdot U_{G_0}$, then all $L_{ub} \subset F_u$. Consider

$$L = \overline{\{L_{ub}\}_{b\in B}};$$

finally, form

$$\widetilde{L} = \overline{\{bLb^{-1}\}_{b\in B}}.$$

F_u is invariant in B; hence $\widetilde{L} \subset F_u$, and $\widetilde{L}$ is invariant in B by construction. Since

$$g_{iLg}i^{-1} = L, \qquad g_{iBg}i^{-1} = B,$$

we have

$$g_i\widetilde{L}g_i^{-1} = \widetilde{L}.$$

To complete the proof it is enough to show that $G/\widetilde{L}$ is a Lie group, and this follows quite easily from the construction of $\widetilde{L}$. Every bicompactly generated locally projectively nilpotent Lie group is nilpotent (see (5)).

Corollary. A bicompactly generated locally projectively nilpotent group is projectively Lie.

Remark. It is well known that the formulated corollary is false for arbitrary bicompactly generated solvable groups. It will be shown below that it also fails for topological N-groups.

§ 2. Recall that an element g of a topological group G is called **bicompact** if $\{g\}$ is a bicompact group. Let B be the set of all bicompact elements of the group G. As shown in (1, 2), in the case of a locally nilpotent and even locally projectively nilpotent group G, the set B turns out to be a closed subgroup, and the factor group G/B is pure, i.e. contains no bicompact elements. One can show that an analogous assertion is also true for topological N-groups (5).

It turns out that the following general result holds:

Theorem 2. Let, in a locally bicompact group G, the set of bicompact elements B form a subgroup. Then the following assertions are valid: 1) B is a closed invariant subgroup; 2) G/B is a pure group; 3) G/N, where $N = B \cdot G_0$, is a discrete torsion-free group, and N is the minimal normal divisor with this property.

Theorem 3. A connected semisimple Lie group with a dense set of bicompact elements is bicompact.

Proof of Theorem 3 is carried out as follows. It is enough to consider the case of a linear semisimple group. For the linear case it is established that the characteristic roots of the matrices must be equal to 1 in modulus. Then, using the classical Cartan–Weyl theorems on the structure of semisimple Lie algebras, we specialize

choice of an additional solvable plane in the Weyl basis. From this and from the equality to zero of the characteristic roots of the matrix of the solvable plane one easily obtains the compactness of the Lie algebra, and hence also of the group.

Remark. For arbitrary, and even solvable, Lie groups Theorem 3 is no longer true. One such example is the semidirect product of the one-dimensional vector group and the torus. It is verified directly that in this group the set of bicompact elements is dense.

Proof of Theorem 2. Suppose first that G is a Lie group. From the Cartan-Mal' cev-Iwasawa theorem ([7,8]) it follows easily that $\overline{B}$ is a connected Lie group. For the proof we use induction on the dimension of $\overline{B}$. If $\overline{B}$ is semisimple, then it is bicompact by Theorem 3. Let now A be a connected abelian closed invariant subgroup ($A \neq e$) of the group $\overline{B}$. Then $A = H \times V$, where H is the bicompact part and V is a vector group. If $H \neq (e)$, then $\overline{B}/H$ is bicompact by the induction hypothesis, whence the bicompactness of $\overline{B}$ follows. If $H = (e)$, then consider $\widetilde{B} = \overline{B}/V$. Relying on a well-known theorem of K. Iwasawa ([8]) on the splitting of a bicompact extension of a vector group, one can show that in $\widetilde{B}$ the set of bicompact elements forms a dense subgroup. Since $\dim \widetilde{B} < \dim \overline{B}$, it follows that $\widetilde{B}$ is bicompact. By the theorem of K. Iwasawa cited above, $\overline{B} = L \cdot V,\ L \cap V = (e)$. If $b \in \overline{B}$, then $b = l \cdot v,\ v = l^{-1} \cdot b = e$, i.e. $\overline{B} = L$. Thus $\overline{B}$ is bicompact; consequently, $\overline{B} = B$. If G is an arbitrary Lie group, then $\overline{B}_0$ is open in $\overline{B}$, hence $B \cap \overline{B}_0 = D$ is dense in $\overline{B}_0$, from which it follows that $\overline{B}_0$ is bicompact. But $\overline{B}_0 \subset B$, and therefore B is also open and $B = \overline{B}$.

Let now G be a locally bicompact group, and H an open subgroup with bicompact factor group H/G_0. By a well-known theorem of H. Yamabe ([10]), the group H is projectively Lie; consequently, the group $\overline{B} \cap H = H_1$ is also projectively Lie, i.e.

$$H_1 = \lim_{\leftarrow}\{H_1^\alpha, \varphi_{\beta\alpha}\},$$

where H_1^α are Lie groups and the kernels of all homomorphisms $\varphi_\alpha : H_1 \to H_1^\alpha$ are bicompact. Hence it is easily deduced that H_1 is a bicompact group and $H_1 \subset B$. All the more, $B_0 = B \cap G_0$ is bicompact. The closedness of B in the zero-dimensional group G is obvious. The reduction to the zero-dimensional case is carried out in a rather simple way if one applies to the group $G^* = G/B_0$ the general Cartan-Mal' cev-Iwasawa theorem from ([9]), after which the periodic part G^* in the corresponding open subgroup is singled out by a direct factor.

Assertions 2) and 3) of Theorem 2 follow easily from the general Cartan-Mal' cev-Iwasawa theorem obtained by the author in ([9]). Let us prove, for example, the first of them; the remaining ones are proved analogously. From the general Cartan-Mal' cev-Iwasawa theorem it follows that $N = B \cdot G_0$ is an open invariant subgroup of G. Let $h^* = h \cdot B$ be a bicompact element in $G^* = G/B$, i.e. $\Delta^* = \overline{\{hB\}}$ is bicompact in G^*. $N^* = N/B \simeq G_0/B_0$, therefore

$$\Delta^* \cdot N^*/N^* \simeq \Delta^*/\Delta^* \cap N^*,$$

i.e. $\Delta^* \cdot N^*/N^*$ is a discrete bicompact, hence finite, group. It follows that $h^k \in N^*$, i.e. $h^k \in N$. h^k is a bicompact element in N^*, but $N^* \simeq G_0/B_0$, hence

N^* is a pure group, and therefore $h^k \in B$, and, as is not difficult to see, $h \in B$, proving the purity of G^*.

§ 3. A topological group Γ is called an N-group if it satisfies the normalizer condition for closed subgroups.

In ([5]) we showed that a locally bicompact N-group (even a Lie N-group) need not be locally projectively nilpotent, i.e. the analogue of B. I. Plotkin' s theorem for abstract N-groups does not hold in the topological case. At the same time, in ([5]) a number of positive results on the structure of topological N-groups were obtained. However, the theory could not be regarded as complete, since the question of local projective nilpotence

N-groups was not solved in the most important case of zero-dimensional N-groups. In addition, the topological analogue remained open of the problem, unsolved in the abstract case: whether every N-group contains a dense ZA-subgroup. In the present paragraph a negative solution of these questions will be given.

Definition. Let F and G be topological groups; to each $f \in F$ assign a copy G_f of the group G, and consider the topological product $\hat{G} = \prod G_f$ in the Tikhonov topology. By X denote the semidirect product of F and $\hat{G}$ with the following defining relation: if $g = (g_f) \in \hat{G}$, $h \in F$, then $hgh^{-1} = g^h = (g_{hf}) \in \hat{G}$. It is easy to verify that X will indeed be an abstract group. Suppose, in addition, that the mapping $\psi : (h, g) \to hgh^{-1}$ is continuous in h and g simultaneously. Then X is a topological group, which we call the **topological wreath product** of F and G, denoted as $X = F \wr G$. The construction just given is a slight modification of the abstract wreath product.

Examples show that the semidirect product of F and $\hat{G}$ will not always be a topological group. However, it is easy to see that X will be a topological wreath product if the group F is considered in the discrete topology.

Let now F be the infinite cyclic group, and G the cyclic group of order two. We note that, without prejudice to the further arguments, G may also be taken to be any finite nilpotent group. Thus $F = \{f\}$, $G = \{g\}$, $g^2 = 1$. Consider $X = F \wr G$. In this case the elements of $\hat{G}$ are indexed by the exponents of the element f. If $\hat{g} = (g_i)$, then $f\hat{g}f^{-1} = \hat{g}' = (g'_i)$, where $g'_i = g_{i+1}$. By an inductive construction it is established that the group X has an infinite number of hypercenters $Z_1, Z_2, \dots, Z_k, \dots$, with $Z_\omega = Z_{\omega+1}$; all hypercenters are finite and belong to $\hat{G}$. Consider the group $\Phi = F \cdot \bar{Z}_\omega$.

Theorem 4. *The group Φ is a topological nilpotent N-group which is not locally projectively nilpotent.*

Proof. Let H be a closed true subgroup of Φ. If $Z_\omega \not\subset H$, then it is easy to prove that $N_\Phi(H) \neq H$; but if $Z_\omega \subset H$, then $\bar{Z}_\omega \subset H$ and $N_\Phi(H) = \Phi$. Thus Φ is a topological N-group. Suppose that Φ is locally projectively nilpotent. Then, by virtue of bicompact generation, by theorem 1 Φ is projectively left. Let U

be such a neighborhood of the identity in Φ that $Z_1 \cap U = (e)$. Then there must exist such a normal divisor $L \subset U$ that Φ/L is a Lie group. Obviously, L is open in Φ. $Z_\omega \cap L = (e)$; otherwise, if i is the least number for which $z_i \in L$, $z_i \in (Z_i \setminus Z_{i-1})$, then $f z_{if}^{-1} z_i^{-1} \in Z_{i-1} \cap L$, and $f z_{if}^{-1} z_i^{-1} \neq (e)$, otherwise $z_i \in Z_1$. $L \setminus (e)$ is an open set and $Z_\omega \cap (L \setminus e) = \emptyset$, which contradicts the density of Z_ω, for one may assume that $U \subset \bar{Z}_\omega$. The contradiction proves theorem 4.

Corollary 1. *There exist locally bicompact zero-dimensional N-groups which do not possess a dense ZA-subgroup.*

Corollary 2. *There exist bicompactly generated zero-dimensional N-groups which are not projectively left.*

Belorussian State University
named after V. I. Lenin

Received
12 XII 1964

REFERENCES

1. V. P. Platonov, DAN, 158, No. 4 (1964).
2. V. M. Glushkov, Tr. Mosk. matem. obshch., 4, 291 (1955).
3. A. Gleason, Duke Math. J., 18, No. 1 (1951).
4. V. M. Glushkov, UMN, 12, issue 2, 3 (1957).
5. V. P. Platonov, Dokl. AN BSSR, 8, No. 12 (1964).
6. L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.
7. A. I. Mal' tsev, Matem. sborn., 58, No. 2 (1945).
8. K. Iwasawa, Ann. Math., 50, No. 3 (1949).
9. V. P. Platonov, DAN, 158, No. 3 (1964).
10. H. Yamabe, Ann. Math., 58, No. 3 (1953).

Note: Figure translations are in progress. See original paper for figures.