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Abstract

Full Text

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AN APPLICATION OF A THEOREM OF A. CARTAN TO MATHEMATICAL STATISTICS

In the theory of analytic sheaves, the theorem (A) of A. Cartan is well known⁽¹⁾.

Let X be an arbitrary K. Stein manifold, and let F be a coherent analytic sheaf over X . Then, for any point $x \in X$, the image of the zero cohomology group $H^0(X, F)$ in F_x generates F_x as an O_x -module.

From this theorem it follows, in particular, that in the ring O_z of functions of complex variables $f(z_1, \dots, z_k)$, holomorphic on a compact simply connected polycylinder, all ideals are finitely generated.

The indicated corollary can be applied to the investigation of phenomena of identical distribution and independence of various kinds of statistics.

Let $(\mathfrak{X}, \mathfrak{A}, \mu)$ be some probability space, and let $Q_1(x)$ and $Q_2(x)$ (for $x \in \mathfrak{X}$) be two scalar statistics, each of which measurably maps \mathfrak{X} onto the abscissa axis with the Borel σ -algebra of measurable sets. Suppose, further, that some family of continuous functions $\{\Phi_k(u)\}$ ($k = 1, 2, \dots$) of a real variable u is given, possessing property (D) with respect to the statistics Q_1 and Q_2 :

- (D) **Determinacy of the analogue of the moment problem.** All mathematical expectations $E_\mu(\Phi_k(Q_i))$ exist ($i = 1, 2; k = 1, 2, \dots$), and their specification determines the distributions of Q_1 and Q_2 up to a μ -measure zero set. We shall call them Φ -moments.

Assume now that the statistics Q_1 and Q_2 have the following property (Γ).

Property (Γ): **“holomorphic connectedness”** of the statistics Q_1 and Q_2 with respect to the family $\{\Phi_k(u)\}$. The statistics Q_1 and Q_2 can be “joined by a path” in the following sense: one can construct a family of statistics $Q(X; a_1, \dots, a_s)$, where (a_1, \dots, a_s) are real parameters such that

$$Q(X; a_1^{(0)}, \dots, a_s^{(0)}) = Q_1(x), \quad Q(X; a_1^{(1)}, \dots, a_s^{(1)}) = Q_2(x),$$

and moreover the Φ -moments

$$E_\mu(\Phi_k(Q(X; a_1, \dots, a_s))) = \varphi_k(a_1, \dots, a_s) \tag{1}$$

exist and are all holomorphic on some compact and simply connected complex polycylinder containing inside it the segment between $(a_1^{(0)}, \dots, a_s^{(0)})$ and $(a_1^{(1)}, \dots, a_s^{(1)})$.

Under this condition, the property of the statistics Q_1 and Q_2 being identically distributed turns out to be “finitely generated by Φ -moments.” More precisely, the following theorem holds:

Theorem 1. There exists a finite integer constant R , depending only on $(\mathfrak{X}, \mathfrak{A}, \mu)$ and on the family $\{\varphi_k = (a_1, \dots, a_s)\}$ ($k = 1, 2, \dots$), such that if the first R Φ -moments of Q_1 and Q_2 coincide,

$$E_\mu(\Phi_k(Q_1)) = E_\mu(\Phi_k(Q_2)) \quad (k = 1, 2, \dots, R), \quad (2)$$

then the statistics Q_1 and Q_2 are identically distributed: $Q_1 \cong Q_2$. The constant R plays the same role for detecting identical distribution of any two statistics $Q(X; a_1, \dots, a_s)$ of the previously constructed family.

To characterize the property of independence of two statistics Q_1 and Q_2 by a “finite number of uncorrelatednesses,” one can propose an analogous theorem.

Introduce two families of continuous functions: $\{\Phi_k(u)\}$ and $\{\Psi_l(u)\}$ such that, from the countable set of equalities

$$E_\mu(\Phi_k(Q_1)\Psi_l(Q_2)) = E_\mu(\Phi_k(Q_1))E_\mu(\Psi_l(Q_2)) \quad (3)$$

($k, l = 1, 2, \dots$), the independence of the statistics Q_1 and Q_2 follows. Further, let the statistics Q_1 and Q_2 possess the property (Γ) of holomorphic connectedness, analogous to the preceding one, in the sense that the functions

$$E_\mu(\Phi_k(Q(X; a_1, \dots, a_s))\Psi_l(Q(X; a'_1, \dots, a'_s))) - E_\mu(\Phi_k(Q(X; a_1, \dots, a_s)))E_\mu(\Psi_l(Q(X; a'_1, \dots, a'_s))) = \gamma_{kl}(a, a') \quad (4)$$

are holomorphic for $k, l = 1, 2, \dots$ on one and the same compact simply connected polycylinder of the complex variables $(a_1, \dots, a_s; a'_1, \dots, a'_s)$, containing the product of the real intervals indicated earlier. Then the following theorem holds:

Theorem 2. There exists a finite integer constant M , depending only on $(\mathfrak{X}, \mathfrak{A}, \mu)$ and on the family $\gamma_{kl}(a, a')$, such that for the independence of the statistics Q_1 and Q_2 it is sufficient that the relations

$$E_\mu(\Phi_k(Q_1)\Psi_l(Q_2)) = E_\mu(\Phi_k(Q_1))E_\mu(\Psi_l(Q_2))$$

hold for k, l not exceeding M .

Thus, in our family the independence of statistics is equivalent to a bounded number of “uncorrelatednesses.”

Theorems 1 and 2 are derived directly from the above-mentioned corollary of theorem (A) of H. Cartan. In the case of theorem 1 we consider the ideal generated by functions of the form

$$\varphi_k(a_1, \dots, a_s) - \varphi_k(a'_1, \dots, a'_s), \quad k = 1, 2, \dots, \quad (5)$$

on the topological product of the previously introduced polycylinders. It has a finite basis consisting of R terms of the indicated form, where R depends only on the objects specified in the assumptions of the theorem. Put

$$(a_1, \dots, a_s) = (a_1^{(0)}, \dots, a_s^{(0)}), \quad (a'_1, \dots, a'_s) = (a_1^{(1)}, \dots, a_s^{(1)}).$$

If, in this case, the first R functions of the form (5) vanish, then all functions of the ideal vanish, and we obtain what was required. Theorem 2 is proved analogously.

Let us give some particular cases of the theorems proved (see ^(2,3)). Let $(\mathfrak{X}, \mathfrak{A}, \mu)$ correspond to a repeated normal sample $X = (x_1, \dots, x_n)$; $x_i \in N(m, \sigma^2)$ (normal law); $Q_1(x)$ and $Q_2(x)$ are polynomial statistics of degrees m_1 and m_2 . The parameters (a_1, \dots, a_s) here may be the coefficients of the polynomials; the “path” joining the statistics consists in a continuous change of the coefficients. Let $G(x)$ be a nonnegative polynomial of degree ≥ 2 . As the family $\{\Phi_k(u)\}$ in theorem 1 one may take the functions $\Phi_k(u) = u^k \exp(-G(u))$. As the families $\{\Phi_k(u)\}$ and $\{\Psi_l(u)\}$ in theorem 2 one may take

$$\Phi_k(u) = u^k \exp(-G_1(u)), \quad \Psi_l(u) = u^l \exp(-G_2(u)),$$

$k, l = 1, 2, \dots$; $G_1(x)$; $G_2(x)$ are nonnegative polynomials of degrees ≥ 2 .

Let $(\mathfrak{X}, \mathfrak{A}, \mu)$ be any Euclidean probabilized space E_n with a Borel σ -algebra. One may take $\Phi_k(u) = u^k \exp(-u^2)$ and, as the family of statistics of theorem 1, polynomials $Q(X; a_1, \dots, a_s)$ under the condition

$$|Q(X; a_1, \dots, a_s)| \geq |x_1|^\varepsilon + \dots + |x_n|^\varepsilon$$

for a_1, \dots, a_s restricted to a real domain, some $\varepsilon > 0$, and $X = (x_1, \dots, x_n) \in E_n$.

There exist many other examples.

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References

¹ H. Cartan, *Variétés analytiques complexes et cohomologie*, Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 11-14, III, 1953, Liège–Paris, 1953.

² Yu. V. Linnik, Bull. Calcutta Math. Soc., 1958-1959, p. 95. ³ A. A. Zinger, Yu. V. Linnik, *Theory of Probability and Its Applications*, 9, no. 3, 547 (1964).

Note: Figure translations are in progress. See original paper for figures.

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