N. V. KUZNETSOV

1965

SovietRxiv

Abstract

Full Text

N. V. KUZNETSOV

ASYMPTOTIC FORMULAS FOR THE EIGENVALUES OF AN ELLIPTIC MEMBRANE

(Presented by Academician A. A. Dorodnitsyn, 30 X 1964)

MATHEMATICS

1. In the present paper the asymptotic distribution of the eigenvalues of the following boundary-value problems is studied:

$$\Delta u + \lambda u = 0 \text{ in } D; \qquad u\big|_\Gamma = 0 \text{ or } \partial u/\partial n\big|_\Gamma = 0. \tag{1,1}$$

Here D is an ellipse with semiaxes $a \operatorname{ch} \xi_0$, $a \operatorname{sh} \xi_0$, $\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. We denote the eigenvalues of problem (1,1), arranged in nondecreasing order, by λ_n $(n = 0, 1, 2, ...)$, and let

$$N(\lambda) = \sum_{\lambda_n \leqslant \lambda} 1. \tag{1,2}$$

Theorem 1. *As* $\lambda \to \infty$,

$$4\pi N(\lambda) = S\lambda \pm L\sqrt{\lambda} + O(\lambda^{1/3}), \tag{1,3}$$

where S *and* L *are the area and the length of the boundary of the ellipse; the upper sign is taken for the boundary condition* $\partial u/\partial n|_\Gamma = 0$, *and the lower one for the condition* $u|_\Gamma = 0$.

Until now, similar asymptotic formulas were known for a membrane having the shape of a rectangle ([1]) and of a circle.

2. We obtain the assertion of Theorem 1 by reducing the problem of the number of eigenvalues of problem (1,1) to the counting of lattice points in certain planar domains (cf. ([2])). To estimate the number of the latter we use van der Corput' s theorem (for the formulation see, for example, ([2]), and for the proof, ([3])).

The central point of the work is the proof of the following proposition.

Theorem 2. *Let, for* $0 \leqslant \alpha \leqslant \operatorname{ch} \xi_0$,

$$c^2(\alpha) = \frac{4}{\pi}\int_0^{\arccos\alpha} \sqrt{\cos^2 t - \alpha^2}\,dt \qquad (c^2(\alpha) \geqslant 0), \tag{2,1}$$

$$\mathcal{E}(\alpha) = \frac{1}{\pi}\int_0^{\pi/2} \operatorname{Re}\sqrt{\alpha^2 - \cos^2 t}\,dt, \qquad E(\alpha) = \frac{1}{\pi}\int_0^{\operatorname{ch}\xi_0} \operatorname{Re}\sqrt{\operatorname{ch}^2 t - \alpha^2}\,dt. \tag{2,2}$$

Put

$$\omega_\pm(\mu,\alpha) = \frac{\mu c^2}{4\pi}\ln\frac{\mu c^2}{4e} + \frac{1}{\pi}\arg\Gamma\left(\frac{1}{2} \mp \frac{1}{4} - \frac{i\mu c^2}{4}\right). \tag{2,3}$$

Let N_+ and N_- be the numbers of lattice points $n \geqslant \frac{1}{2}$, $m \geqslant \frac{1}{2}$ under the curves $m_+ = m_+(n_+)$, $m_- = m_-(n_-)$, specified by the equations

$$n_\pm(\mu,\alpha) = 2\mu\mathcal{E}(\alpha) + 2\operatorname{sign}(\alpha-1)\omega_\pm(\mu,\alpha) + \tfrac{1}{2} \pm \tfrac{1}{4} + O(1/\sqrt{\mu}); \tag{2,4}$$

$$m_\pm(\mu,\alpha) = \mu E(\alpha) + \operatorname{sign}(1-\alpha)\omega_\pm(\mu,\alpha) + \tfrac{1}{4} \pm \tfrac{1}{8} + O(1/\sqrt{\mu}) + O(1/\sqrt{\mu E}). \tag{2,5}$$

Then the number of eigenvalues of problem (1,1) not exceeding μ^2, under the boundary condition $u|_\Gamma = 0$, is equal to the sum $N_+ + N_-$.

We note that equations (2.6)–(2.13), by which the boundaries of the domains are given, are uniform in α. To obtain these equations we shall have to investigate the asymptotic behavior of the solutions of Mathieu' s equation; the results are set forth in the lemmas below.

3. The variables in (1.1) are separated if one sets $x_1 + ix_2 = a\operatorname{ch}(\xi + i\eta)$, $0 \le \xi \le \xi_0$, $0 \le \eta < 2\pi$. We write the separated equations in the form

$$\frac{d^2}{d\xi^2}F(\xi) + \mu^2(\operatorname{ch}^2\xi - \alpha^2)F(\xi) = 0; \tag{3.1}$$

$$\frac{d^2}{d\eta^2}G(\eta) + \mu^2(\alpha^2 - \cos^2\eta)G(\eta) = 0. \tag{3.2}$$

Here $\mu^2\alpha^2$ is the separation parameter, and μ is connected with the spectral parameter λ by the equality $\mu^2 = \lambda a^2$. Below we denote by $\operatorname{Ai}(z)$ the Airy function (see, for example, ([4])) and by $U(a,z)$ the Weber function normalized as in ([5],[6]); $\delta > 0$ is a sufficiently small fixed number. In Lemmas 1-4, $0 \le \eta \le \pi/2$; for obtaining equations for the eigenvalues this is sufficient, since the periodicity condition for even and odd solutions of equation (3.2) is equivalent to the requirement

$$\frac{d}{d\eta}G^2(\eta)\Big|_{\eta=\pi/2} = 0, \tag{3.3}$$

which uses knowledge of the solution only on the interval $[0, \pi/2]$; for more detail on the boundary conditions for equations (3.1)–(3.2), see ([7]).

Lemma 1. Let $0 \le \alpha \le \delta$, and let the functions $c^2(\alpha)$ and $x(\eta)$ be defined by the equalities

$$c^2(\alpha) = \frac{4}{\pi}\int_{\arccos\alpha}^{\pi/2} \sqrt{\alpha^2 - \cos^2 t}\, dt; \tag{3.4}$$

$$\frac{1}{2}x\sqrt{c^2 - x^2} - \frac{1}{2}c^2 \arccos\frac{x}{c} + \frac{\pi c^2}{2} = \int_{\arccos\alpha}^{\eta} \sqrt{\alpha^2 - \cos^2 t}\, dt. \tag{3.5}$$

Put

$$R_1(x) = \frac{2}{\sqrt{\mu}}\int_x^{x(\pi/2)} \frac{\left|x^{1/4}\dfrac{d^2}{dx^2}x^{-1/4}\right| dx}{\left(1 + \mu^{1/12}c^{1/6} + \mu^{1/4}|x^2 - c^2|^{1/4}\right)^2}, \qquad R_2 = R_1(x(0)) - R_1(x). \tag{3.6}$$

Equation (3.2) has solutions G_1 and G_2 such that, as $\mu \to \infty$,

$$G_1 = \dot{x}^{-1/2}U(\mu c^2/2, e^{i\pi/4}\sqrt{2\mu}\, x)\,[1 + O(e^{R_1} - 1)]; \tag{3.7}$$

$$G_2 = \dot{x}^{-1/2}U(\mu c^2/2, e^{-i\pi/4}\sqrt{2\mu}\, x)\,[1 + O(e^{R_2} - 1)]. \tag{3.8}$$

Lemma 2. Let $\alpha \in [\delta, 1-\delta]$, and let $x(\eta)$ be defined by the equation

$$\frac{2}{3}x^{3/2} = \int_{\arccos\alpha}^{\eta} \sqrt{\alpha^2 - \cos^2 t}\, dt; \tag{3.9}$$

$$R_3(x) = \frac{2}{\mu^{2/3}}\int_{x(0)}^{x} \frac{\left|x^{1/4}\dfrac{d^2}{dx^2}x^{-1/4}\right| dx}{\left(1 + \mu^{1/6}|x|^{1/4}\right)^2}, \qquad R_4(x) = R_3(x(\pi/2)) - R_3(x). \tag{3.10}$$

Equation (3.2) has solutions G_1 and G_2 such that, as $\mu \to \infty$,

$$G_1 = \dot{x}^{-1/2} \left\{ \mathrm{Ai}(-\mu^{2/3} x) + \frac{\left|\exp\left(-\frac{2}{3}\mu(-x)^{3/2}\right)\right|}{1 + \mu^{1/6}|x|^{1/4}} \, O(e^{R_3} - 1) \right\}; \qquad (3.11)$$

$$G_2 = \dot{x}^{-1/2} \left\{ \mathrm{Ai}(-\mu^{2/3} e^{2i\pi/3} x) + \frac{\left|\exp\left(\frac{2}{3}\mu(-x)^{3/2}\right)\right|}{1 + \mu^{1/6}|x|^{1/4}} \, O(e^{R_4} - 1) \right\}. \qquad (3.12)$$

Lemma 3. Let $\alpha \in [1 - \delta, 1]$,

$$c^2(\alpha) = \frac{4}{\pi} \int_0^{\arccos \alpha} \sqrt{\cos^2 t - \alpha^2} \, dt$$

and

$$\varphi(x) = \frac{1}{2} x \sqrt{x^2 - c^2} - \frac{1}{2} c^2 \ln\left(\frac{x}{c} + \sqrt{\frac{x^2}{c^2} - 1}\right) = \int_{\arccos \alpha}^{\eta} \sqrt{\alpha^2 - \cos^2 t} \, dt. \qquad (3,13)$$

Put

$$R_5(x) = \frac{2}{\sqrt{\mu}} \int_0^x \frac{\left| x^{1/4} \dfrac{d^2}{dx^2} x^{-1/4} \right| dx}{\left(1 + \mu^{1/12} c^{1/6} + \mu^{1/4} |x^2 - c^2|^{1/4}\right)^2}, \qquad R_6(x) = R_5(x(\pi/2)) - R_5(x). \qquad (3,14)$$

There exist solutions G_1 and G_2 of equation (3,2) such that, as $\mu \to \infty$,

$$G_1 = x^{-1/2} \left\{ U\left(\frac{i\mu c^2}{2}, \, e^{-i\pi/4}\sqrt{2\mu x}\right) + \frac{\left|\exp\left(-\frac{1}{8}\pi\mu c^2 + i\mu c^2 \varphi(x)\right)\right|}{1 + (\mu c^2)^{1/12} + \mu^{1/4}|x^2 - c^2|^{1/4}} \, O\left(e^{R_5} - 1\right) \right\}; \qquad (3,15)$$

$$G_2 = x^{-1/2} \left\{ U\left(-\frac{i\mu c^2}{2}, \, e^{i\pi/4}\sqrt{2\mu x}\right) + \frac{\left|\exp\left(-\frac{\pi\mu c^2}{8} - i\mu c^2 \varphi(x)\right)\right|}{1 + (\mu c^2)^{1/12} + \mu^{1/4}|x^2 - c^2|^{1/4}} \, O\left(e^{R_6} - 1\right) \right\}. \qquad (3,16)$$

Lemma 4. Let $1 \le \alpha \le 1 + \delta$,

$$c^2(\alpha) = \frac{4}{\pi}\int_0^{\operatorname{arch}\alpha} \sqrt{\alpha^2 - \operatorname{ch}^2 t}\, dt$$

and

$$\frac{1}{2}x\sqrt{x^2+c^2} + \frac{c^2}{2}\ln\left[\frac{x}{c} + \sqrt{\frac{x^2}{c^2}+1}\right] = \int_0^{\eta} \sqrt{\alpha^2 - \cos^2 t}\, dt. \qquad (3{,}17)$$

Put

$$R_7(x) = \frac{2}{\sqrt{\mu}}\int_0^x \frac{\left|x^{1/4}\frac{d^2}{dx^2}x^{-1/4}\right| dx}{\left(1+(\mu c^2)^{1/12}+\mu^{1/4}|x^2-c^2|^{1/4}\right)^2}, \qquad R_8 = R_7(x(\pi/2)) - R_7(x). \qquad (3{,}18)$$

Equation (3,2) has solutions G_1 and G_2 such that, as $\mu \to \infty$,

$$G_1 = x^{-1/2}\left\{U\left(-\frac{i\mu c^2}{2},\, e^{-i\pi/4}\sqrt{2\mu x}\right) + \frac{e^{-\frac{1}{8}\pi\mu c^2}}{1+(\mu c^2)^{1/12}+\mu^{1/4}|x^2-c^2|^{1/4}}\, O\left(e^{R_7}-1\right)\right\}; \qquad (3{,}19)$$

$$G_2 = x^{-1/2}\left\{U\left(\frac{i\mu c^2}{2},\, e^{i\pi/4}\sqrt{2\mu x}\right) + \frac{e^{-\frac{1}{8}\pi\mu c^2}}{1+(\mu c^2)^{1/12}+\mu^{1/4}|x^2-c^2|^{1/4}}\, O\left(e^{R_8}-1\right)\right\}. \qquad (3{,}20)$$

Lemma 5. For $1-\delta \le \alpha \le 1$ and $\mu \to \infty$, equation (3,1) has solutions $F_1(\xi)$ and $F_2(\xi)$ for which the asymptotic formulas are the same as in Lemma 4, provided only that by $c^2(\alpha)$ one understands

$$\frac{4}{\pi}\int_0^{\arccos\alpha} \sqrt{\cos^2 t - \alpha^2}\, dt$$

and $x(\eta)$ is replaced by $y(\xi)$, defined by the equality

$$\frac{1}{2}y\sqrt{y^2+c^2} + \frac{c^2}{2}\ln\left[\frac{y}{c} + \sqrt{\frac{y^2}{c^2}+1}\right] = \int_0^{\xi} \sqrt{\operatorname{ch}^2 t - \alpha^2}\, dt. \qquad (3{,}21)$$

Lemma 6. For $1 \le \alpha \le 1+\delta$ and $\mu \to \infty$, equation (3.1) has solutions $F_1(\xi)$ and $F_2(\xi)$ for which the asymptotic formulas are the same as in Lemma 3, provided that $c^2(\alpha)$ is understood to mean

$$\frac{4}{\pi}\int_0^{\operatorname{ar\,ch}\alpha}\sqrt{\alpha^2-\operatorname{ch}^2 t}\,dt$$

and $x(\eta)$ is replaced by the function $y(\xi)$, defined by the equation

$$\frac{1}{2}y\sqrt{y^2-c^2}-\frac{c^2}{2}\ln\left[\frac{y}{c}+\sqrt{\frac{y^2}{c^2}-1}\right]=\int_{\operatorname{ar\,ch}\alpha}^{\xi}\sqrt{\operatorname{ch}^2 t-\alpha^2}\,dt. \qquad (3,22)$$

Lemma 7. For $\alpha \geq 1+\delta$, the formal replacement of $x(\eta)$ in Lemma 2 by

$$\left[\frac{3}{2}\int_{\operatorname{ar\,ch}\alpha}^{\xi}\sqrt{\operatorname{ch}^2 t-\alpha^2}\,dt\right]^{2/3}$$

gives an asymptotic estimate for the solutions of equation (3.1).

We note that in all the lemmas the symbols O may be replaced by $O(1/\sqrt{\mu})$, uniformly in η, ξ and uniformly in α. Estimates for the derivatives can be obtained by termwise differentiation of the asymptotic formulas presented above. The estimates of the solutions of equation (3.1) for $\alpha \leq 1-\delta$ and of equation (3.2) for $\alpha \geq 1+\delta$ are trivial, and we do not write out the corresponding formulas here.

To prove Theorem 2 it now suffices to use the boundary conditions for equations (3.1)–(3.2) and the asymptotic expansions of the functions $\operatorname{Ai}(z)$ ([4]) and $U(a,z)$ ([6,8]). Application of the Van der Corput theorem to each of the regions (2.6) –(2.13) completes the proof of Theorem 1. The arguments for the boundary condition $\partial u/\partial n|_\Gamma = 0$ do not differ in any essential way from those listed above.

4. We formulate, in conclusion, a result following from the estimates in Lemmas 1–7.

Theorem 3. Let D be a domain in the two-dimensional (x, y)-plane, bounded by arcs of confocal ellipses and hyperbolas:

$$x+iy=a\operatorname{ch}(\xi+i\eta), \qquad (4,1)$$

where $0 \leq \xi_0 \leq \xi \leq \xi_1, \quad 0 \leq \eta_0 \leq \eta \leq \eta_1 \leq 2\pi$, with fixed ξ_k, η_k $(k=0,1)$. Then Theorem 1 is valid for D.

The author expresses his gratitude to Prof. V. B. Lidskii for his constant attention to the work.

Moscow Institute of Physics and Technology

Received
25 X 1964

CITED LITERATURE

1. F. H. Brownell, *Pacific J. Math.*, **5**, No. 4 (1955).
2. E. Titchmarsh, *Proc. Roy. Soc.*, **A252**, No. 1271 (1959).
3. E. Landau, *Vorlesungen über Zahlentheorie*, 2, Leipzig, 1927.
4. F. Olver, *Phil. Trans. Roy. Soc.*, **A247**, No. 931 (1954).
5. F. Olver, *Proc. Cambridge Phil. Soc.*, **57**, No. 4 (1961).
6. F. Olver, *J. Res. Nat. Bur. Stand.*, **B63**, No. 2 (1959).
7. J. Meixner, F. Schäfke, *Mathieusche und Spheroidalfunktionen*, Berlin, 1954.
8. E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, 2, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.