ON PERIODIC REGIMES OF SOME DISTRIBUTED SYSTEMS

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Abstract

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Self-oscillatory processes that occur under certain conditions in electrolytic systems were considered in works ([1–3]). In work ([4]), a periodic solution of the corresponding nonlinear problem was obtained under the assumption that the oscillograms $i = i(t)$ found experimentally are known (here i is the current density, t is time). Below, an explicit periodic solution is found for the diffusion equation with a certain nonperiodic boundary condition.

The theory of self-oscillations occurring under certain conditions in electrochemical systems and considered in works ([1–4]) proceeds from the assumptions that the characteristic of the system has a descending segment and that the transfer of the discharging substance from the solution to the electrode surface is a slow diffusion process.

Let $c(x,t)$ denote the concentration of the substance in the electrolyte solution. The function $c(x,t)$ satisfies the diffusion equation with diffusion coefficient D under certain initial and boundary conditions.

Introduce the reduced concentration $u(x,t)$ by the formula

$$u(x,t) = c(x,t) - c_0 - (c^0 - c_0)x/l. \tag{1}$$

Here $c_0 + (c^0 - c_0)x/l$ is the stationary state of the system; c_0 is its value $c(0,t)$ at the electrode surface; l is the thickness of the diffusion layer; c^0 is the value of the concentration at the boundary $x = l$ ([1–3]).

If the stationary state is unique, then under definite conditions self-oscillations are excited in the system ([3]). It is assumed that the capacitance of the double electric layer may be neglected. In this case the reduced concentration $u(x,t)$ is a bounded periodic function satisfying (as may be assumed in a number of cases, in the region $0 \le x < \infty$) the diffusion equation ([1])

$$\partial u/\partial t = D\,\partial^2 u/\partial x^2 \tag{2}$$

with the boundary condition at the electrode surface

$$\partial u/\partial x\big|_{x=0} = F. \tag{3}$$

Here F is a certain S-shaped function depending on the parameters and characteristics of the electrolytic system. In this case the oscillation consists of two stages, to which correspond the branches $F_1(u)$ and $F_2(u)$ of the curve $F(u)$ ([1–4]).

Often, for practical purposes, it is sufficient to approximate these branches as follows:

$$F_i[u(0,t),\, \partial u(0,t)/\partial t] = hu(0,t) + Q_i$$

$$\text{for} \quad (-1)^i \partial u(0,t)/\partial t < 0 \quad (i = 1, 2), \tag{4}$$

where $h > 0$, Q_1, Q_2 are constants.

The reduced concentration $u(x,t)$, defined by formula (1), is chosen so that the condition

$$\lim_{x\to\infty} u(x,t) = 0. \tag{5}$$

is satisfied.

If we assume that for $\alpha < t < \beta$, $\partial u(0,t)/\partial t < 0$, and for $\beta < t < \gamma$, $\partial u(0,t)/\partial t > 0$ ($\alpha = {}^1/_2 pT,\ \beta = -{}^1/_2 pT + T,\ \gamma = {}^1/_2 pT + T,\ 0 < p < 1$), where T is the period of self-oscillation and p is the ratio of the duration of the high-current stage to the oscillation period T, then the constants Q_1 and Q_2, by virtue of condition (5), must be related by

$$Q_1 p + Q_2(1-p) = 0 \qquad (Q_1 - Q_2 > 0). \tag{6}$$

It can be shown that the solution of equation (2) with the boundary condition

$$\partial u/\partial x - hu\,\big|_{x=0} = \chi(t). \tag{7}$$

without initial conditions, when (5) is satisfied, will take the form

$$u(x,t) = -\left(\frac{D}{\pi}\right)^{1/2} \int_{-\infty}^{t} \left\{ \exp\left[-\frac{x^2}{4D(t-\tau)}\right] - \right.$$

$$-h\int_0^\infty \exp\left[-h\xi - \frac{(x+\xi)^2}{4D(t-\tau)}\right] d\xi\Bigg\}\, \frac{\chi(\tau)\, d\tau}{\sqrt{t-\tau}}. \tag{8}$$

We shall now assume that the function $\chi(\tau)$ is determined by the dependence

$$\chi(\tau) = \begin{cases} Q_2 & \text{for } \alpha + kT \le t \le \beta + kT, \\ Q_1 & \text{for } \beta + kT \le t \le \gamma + kT, \end{cases} \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{9}$$

Then, using condition (6), analogously to how this was done in paper (4), we obtain an expression for the function $u_1(0,t)$ on the interval $\alpha \le t \le \beta$ and for the function $u_2(0,t)$ on the interval $\beta \le t \le \gamma$:

$$u_1(0,t) = -\left(\frac{D}{\pi}\right)^{1/2} \left[Q_2 \int_\alpha^t \xi(t-\tau, \theta)\, d\tau + S(t)\right],$$

$$u_2(0,t) = -\left(\frac{D}{\pi}\right)^{1/2} \left[Q_1 \int_\beta^t \xi(t-\tau, \theta)\, d\tau + Q_2 \int_\alpha^\beta \xi(t-\tau, \theta)\, d\tau + S(t)\right]. \tag{10}$$

Here the following notation has been introduced:

$$\xi(y, \theta) = \theta\omega(\theta^2 y), \qquad \theta = h\sqrt{D}, \qquad \omega(z) = \frac{1}{\sqrt{z}} - \sqrt{\pi}\exp z\,[1 - \Phi(\sqrt{z})], \tag{11}$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\eta^2)\, d\eta,$$

$$S(t) = \sum_{k=0}^{-\infty} \left[Q_1 \int_{\beta-T+kT}^{\alpha+kT} \xi(t-\tau, \theta)\, d\tau + Q_2 \int_{\alpha-T+kT}^{\beta-T+kT} \xi(t-\tau, \theta)\, d\tau\right].$$

Figure 1 presents the graph of the function $\omega(z)$, defined by formula (11). From the behavior of the function $\omega(z)$ it follows that, when the condition is satisfied

(6) the alternating series (10) converge, and the inequalities hold:

$$du_1(0,t)/dt > 0, \qquad du_2(0,t)/dt < 0.$$

In paper (2) it was shown that, in order to estimate the period T of self-oscillations of a given electrolytic system with a nonlinear boundary condition of type (3), it is important to consider self-oscillations of rectangular form, occurring with certain prescribed amplitudes. Such self-oscillations correspond to

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

the case when $h = 0$ in condition (4). A periodic solution of equation (2) with condition (3), without initial conditions, was obtained for this case in papers (2, 6). Passing in formulas (11) to the limit as $0 \to 0$, we obtain

$$\xi(y, 0) = y^{-1/2}. \tag{12}$$

Fig. 1

Fig. 2

For a number of values of the parameter p, formulas (10) and (11), using notation (12), were used to compute the concentration $u(0, t)$ on the high-speed BESM-2 computer. Figure 2 presents the dependence of the dimensionless function $U(\tau, p)$ on the dimensionless variable τ. Here

$$U(\tau, p) = \frac{u(0,t)}{(Q_1 - Q_2)\sqrt{DT}}, \qquad \tau = \frac{t}{T}. \tag{13}$$

The dependence of $u(0, t)$ on t for specific values of the quantities D, T, $Q_1 - Q_2$, as follows from the theory of dimensions (5), is easily found according to formulas (13), where U is a universal function of p and τ.

Let us now set ourselves the task of finding, in the semi-infinite domain $x \geq 0$, a periodic solution of the equation

$$\partial u/\partial t + A\,\partial u/\partial x = -\delta u + \nu\,\partial^2 u/\partial x^2 \tag{14}$$

with boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \begin{cases} h_1 u(0,t) + q_1, & kT \leq t \leq kT + t_1, \\ h_2 u(0,t) + q_2, & kT + t_1 \leq t \leq (k+1)T, \end{cases} \qquad (k = 0, \pm 1, \pm 2, ...) \tag{15}$$

where $h_1 > 0$, $h_2 > 0$, q_1, q_2, A, δ, $\nu > 0$ are constants.

A periodic solution of equation (14) is given by formulas (5.5) of paper (6), where the notation (5.3), (5.4) of paper (6) and (2.2) of paper (7) is used.

It can be shown that the solution of the problem under consideration reduces to finding the constants C_k and D_k $(k = 1, 2, ...)$ from an infinite system of linear equations

$$C_k = \sum_{m=1}^{\infty} (\alpha_{m,k} C_m + \beta_{m,k} D_m) + U_k,$$

$$D_k = \sum_{m=1}^{\infty} (\gamma_{m,k} C_m + \delta_{m,k} D_m) + V_k, \qquad (k = 1, 2, ...) \tag{16}$$

where the following notation has been introduced:

$$r_k = a + \vartheta \rho_k - h_2, \qquad s_k = \vartheta \omega_k, \qquad \Delta_k = r_k^2 + s_k^2,$$

$$v_{k,m} = \frac{(h_1 - h_2)}{2\pi} \left(\frac{k}{m}\right)^{\chi} \left(0 < \chi \leq \frac{1}{4}\right), \tag{17}$$

$$\alpha_{m,k} = v_{k,m}(r_k g_{k,m} + s_k f_{k,m}), \qquad \beta_{m,k} = v_{k,m}(r_k h_{k,m} + s_k e_{k,m}),$$

$$\gamma_{m,k} = v_{k,m}(r_k f_{k,m} - s_k g_{k,m}), \qquad \delta_{m,k} = v_{k,m}(r_k e_{k,m} - s_k h_{k,m}),$$

$$g_{k,m} = \frac{\sin(m-k)\tau_1}{m-k} + \frac{\sin(m+k)\tau_1}{m+k} + \eta \frac{\sin k\tau_1 \sin m\tau_1}{km},$$

$$f_{k,m} = \frac{1 - \cos(k-m)\tau_1}{k-m} + \frac{1 - \cos(k+m)\tau_1}{k+m} + \eta \frac{(1 - \cos k\tau_1) \sin m\tau_1}{km},$$

$$h_{k,m} = \frac{1 - \cos(m-k)\tau_1}{m-k} + \frac{1 - \cos(m+k)\tau_1}{m+k} + \eta \frac{\sin k\tau_1 (1 - \cos m\tau_1)}{km}, \tag{18}$$

$$e_{k,m} = \frac{\sin(m-k)\tau_1}{m-k} - \frac{\sin(m+k)\tau_1}{m+k} + \eta \frac{(1 - \cos k\tau_1)(1 - \cos m\tau_1)}{km},$$

$$\tau_1 = \frac{2\pi t_1}{T}, \qquad \eta = -\frac{1}{\pi} \frac{h_1 - h_2}{\sqrt{a^2 + \delta/\nu} - a + [h_1 \tau_1 + h_2(2\pi - \tau_1)]/2\pi},$$

$$U_k = \frac{k^{\chi-1}}{\pi \Delta_k} \xi \{r_k \sin k\tau_1 + s_k (1 - \cos k\tau_1)\},$$

$$V_k = \frac{k^{\chi-1}}{\pi \Delta_k} \xi \{r_k (1 - \cos k\tau_1) - s_k \sin k\tau_1\}, \tag{19}$$

$$\xi = \frac{1}{2}\left[q_1\tau_1 + q_2(2\pi - \tau_1)\right] + q_1 - q_2.$$

It is easy to see that the infinite system of linear equations (16) is completely regular (8), provided only that the quantity $|h_1 - h_2|$ is sufficiently small. Since the constants U_k and V_k $(k = 1, 2, ...)$, defined by formulas (19), are bounded, the system (16) has a unique bounded solution, which can be found by the method of successive approximations with zero initial conditions (8).

Setting in formulas (14)–(19) $\nu = D,\ A = a = \delta = 0,\ h_1 = h_2 = h$, we obtain the solution of the problem considered above in the form of a Fourier series.

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Note: Figure translations are in progress. See original paper for figures.