Physics

Academician Ya. B. ZEL'DOVICH

1965

SovietRxiv

Abstract

Full Text

Physics

Academician Ya. B. ZEL' DOVICH

THE NUMBER OF QUANTA AS AN INVARIANT OF A CLASSICAL ELECTROMAGNETIC FIELD

Let an electromagnetic field $\mathbf{E}(\mathbf{x}), \mathbf{H}(\mathbf{x})$ be given at $t = \text{const}$, in the absence of charges and currents. Such a field may be regarded as a collection of electromagnetic waves and, by carrying out its Fourier expansion, one can easily find the number of quanta of each frequency and the total number of quanta N of the given field. In the present note an expression is given for N in terms of the coordinate representation of $\mathbf{E}$ and $\mathbf{H}$

$$c\hbar N = \frac{1}{2}\frac{1}{(2\pi)^3}\iint \frac{\mathbf{E}(\mathbf{x})\mathbf{E}(\mathbf{y}) + \mathbf{H}(\mathbf{x})\mathbf{H}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\, d^3x\, d^3y, \tag{1}$$

constructed in the manner of the expressions for the energy and momentum of the field

$$(8\pi)^{-1}\int (E^2 + H^2)\, d^3x, \qquad (4\pi c)^{-1}\int (\mathbf{E}\times\mathbf{H})\, d^3x.$$

The field is expressed in the Gaussian system of units.

Expression (1) is of interest because the right-hand side, being completely defined for a classical field, contains no Planck constant. It is obvious that this classical quantity is an integral of motion of the free field and a relativistic invariant.

One may point to an analogy with an oscillator: the number of the quantum level is related to the energy and frequency by

$$\hbar n = E/\omega. \tag{2}$$

The right-hand side of (2) is an adiabatic invariant; this property can be derived from classical mechanics, but for the modern physicist it is simpler to regard E/ω (or, in the general case, $\int p\, dq$) as a quantity proportional to the number of the state. From this formulation there follows at once not only the adiabatic invariance of E/ω, but also an estimate of the change of E/ω at a finite rate of change of ω, and parametric resonance, i.e. the change $\Delta n = \pm 2$ under a symmetric action of double frequency, etc.

In exactly the same way, from quantum considerations there follows at once the relativistic invariance of the classical right-hand side of (1), which is not easy to prove directly.

The proof of formula (1) is carried out schematically as follows: we decompose $\mathbf{E}$ and $\mathbf{H}$ into transverse waves. For a given $\mathbf{k}$ we introduce two unit vectors $\mathbf{n}_1, \mathbf{n}_2$, perpendicular to $\mathbf{k}$; the third $\mathbf{n}_3 = \mathbf{k}/|\mathbf{k}|$ is directed along $\mathbf{k}$.

The amplitude of two plane-polarized waves traveling in the direction $\mathbf{n}$:

$$C_{k_1} \sim (E_{k_1} + H_{k_2}), \qquad C_{k_2} \sim (E_{k_2} - H_{k_1}); \tag{3}$$

$$E_{k_1} \sim \int (\mathbf{E}\mathbf{n}_1) e^{-i\mathbf{k}\mathbf{x}}\, d^3x. \tag{4}$$

In this case

$$E_{k_3} = H_{k_3} = 0. \tag{5}$$

by virtue of the transversality of the waves, and specifically as a consequence of the conditions

$$\operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0; \tag{6}$$

$$N \sim \int \frac{|C_{k_1}|^2 + |C_{k_2}|^2}{c\hbar |k|}\, d^3k. \tag{7}$$

We substitute expressions (3) and (4) into (7). We take into account that $E_{k_1} = E^*_{-k_1}$, and add under the integral an expression equal to zero,

$$E_{k_3} E^*_{k_3} + H_{k_3} H^*_{k_3} \equiv 0.$$

After this, in the triple integral over x, y, k,

$$\iiint [\mathbf{E}(\mathbf{x})\mathbf{E}^*(\mathbf{y}) + \mathbf{H}(\mathbf{x})\mathbf{H}(\mathbf{y})]\, \frac{1}{k}\, e^{-ik(x-y)}\, d^3k\, d^3x\, d^3y$$

we first integrate with respect to d^3k, which gives (1). Obviously, the same result would be obtained also for an expansion in any other system of waves (circular polarization, electric and magnetic multipoles).

To explain (1), let us introduce $\mathbf{r} = \mathbf{y} - \mathbf{x}$ and rewrite (1) in the form

$$c\hbar N = \frac{1}{16\pi^3}\int d^3x\,\mathbf{E}(\mathbf{x})\left(\int \mathbf{E}(\mathbf{x}+\mathbf{r})\,\frac{1}{r^2}\,d\Omega\,r^2dr\right) + \ldots =$$

$$= \frac{1}{16\pi^3}\int d^3x\,|\mathbf{E}(\mathbf{x})|^2\bar{\lambda} + \ldots \simeq \varepsilon\bar{\lambda}, \qquad \varepsilon = \int \frac{E^2+H^2}{8\pi}\,d^3x, \tag{8}$$

where the ellipsis refers to $\mathbf{H}$; $\bar{\lambda}$ is the mean length over which the correlation between the field at a given point and in its neighborhood extends, i.e., the mean wavelength of the field; ε is the total energy of the field.

It is curious that an expression whose structure is similar to (1) was given by Wheeler [1] in connection with the question of the probability of a given distribution $H(x)$ in the ground state of the vacuum,

$$\Psi(\mathbf{H}) = A\exp\left(-\iint \frac{H(x)H(y)\,d^3x\,d^3y}{16\pi^3\hbar c\,|x-y|^2}\right). \tag{9}$$

I take this opportunity to gratefully note discussions with L. P. Pitaevskii, B. Ya. Zeldovich, and especially A. S. Kompaneets, without whose help this note would not have arisen.

Received4 VI 1965

REFERENCES

[1] B. R. Harrison, R. S. Thorne, M. Wakano, J. A. Wheeler, *Gravitation Theory and Gravitational Collapse*, Chicago, 1965, p. 142.

Note: Figure translations are in progress. See original paper for figures.