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Abstract

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A CRITERION FOR THE REPRESENTABILITY OF EVENTS IN FINITE PROBABILISTIC AUTOMATA

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In the work of M. O. Rabin "Probabilistic Automata" (^{1}) it is shown that the class of events representable in finite probabilistic automata by a set of states (in this paper we shall call them simply representable) is broader than the class of regular events. The method consists in the fact that the cardinality of the set of all representable events is found, and it turns out to be equal to the cardinality of the continuum. From this, in particular, it follows that it is impossible to construct a constructive criterion for characterizing the class of representable events of the kind that we have for the class of regular events. A criterion in one form or another must contain "discontinuity." The criterion given below for the representability of events in finite probabilistic automata is formulated in the language of finite-dimensional vector functions.

A probabilistic automaton is an object

$$A\langle X,\, Y,\, A,\, \mu(a),\, \mu(a', y/a, x)\rangle,$$

where X, Y, A are, respectively, the sets of input symbols, output symbols, and the set of states; $\mu(a)$ is the probability distribution of the initial states of the automaton; $\mu(a', y/a, x)$ is the conditional probability distribution describing the functioning of the automaton. X and Y are finite or countable sets, while the set A is finite in the present paper. Put

$$\mu(a_1 a_2 \dots a_n y_1 y_2 \dots y_n / x_1 x_2 \dots x_n) = \sum_{a_0} \mu(a_0) \prod_{i=1}^{n} \mu(a_i y_i / a_{i-1}, x_i),$$

and further

$$\mu(\cdot a / x_1 x_2 \dots x_n) = \sum_{\substack{a_1 \dots a_{n-1} \\ y_1 \dots y_n}} \mu(a_1 a_2 \dots a_{n-1} a y_1 \dots y_n / x_1 x_2 \dots x_n).$$

Definition (M. O. Rabin). An event S is representable in a finite probabilistic automaton A with initial distribution $\mu(a)$, set of states M, and constant λ, if for words from S, and only for them, the inequality

$$\sum_{a\in M} \mu(\cdot a/p) > \lambda$$

holds.

In what follows we shall consider the stochastic vectors

$$\bar{\mu}(p) = \{\mu(\cdot a_1/p),\ \mu(\cdot a_2/p),\ \dots,\ \mu(\cdot a_N/p)\}, \quad p \in F_x,$$

assuming that $\mu(\cdot a/e) = \mu(a)$, where e is the empty word, and F_x is the free semigroup with identity over the alphabet X. $\bar{\mu}(p)$ is a certain vector function defined everywhere on the free semigroup F_x. We shall call a function $\varphi(p)$, $p \in F_x$, finite-automaton if it is isomorphic

a finite-dimensional vector function of the form $\{\mu(p),\ p \in F_x\}$, where $\mu_i(p) \geq 0$, $i = 1, 2, \dots, N$, $\sum_{i=1}^{N} \mu_i(p) = 1$, and there exist stochastic matrices $A(x_1), A(x_2), \dots, A(x_m), \dots$ such that

$$\mu(x_{i_1} x_{i_2} \dots x_{i_s}) = \mu(e) A(x_{i_1}) \times A(x_{i_2}) \dots A(x_{i_s}).$$

We shall say that a vector $\mathbf{n}$ is **regular** if its coordinates are equal either to zero or to one.

In order that an event S be representable in a finite probabilistic automaton, it is necessary and sufficient that there exist a finite-automaton vector function $z(p)$, $p \in F_x$, $S \subset F_x$, and a regular vector $\mathbf{n}$ such that, for some vector z_0, the equivalence

$$p \in S \sim (z(p) - z_0) \cdot \mathbf{n} > 0$$

holds.

Theorem 1. *In order that a multivalued mapping $\{z(p),\ p \in F_x\}$ of finite-dimensional stochastic vectors define a finite-automaton vector function, it is necessary and sufficient that the following conditions be satisfied:*

1. *For any finite set of words $p_1, p_2, \dots, p_s$ and numbers $\alpha_1, \alpha_2, \dots, \alpha_s$, $\sum_{i=1}^{s} \alpha_i = 0$, the equivalence*

$$\sum_{i=1}^{s} \alpha_i z(p_i) = 0 \sim (r) \quad (r) \in F_x \to \sum_{i=1}^{s} \alpha_i z(p_i r) = 0$$

   holds.

2. *If $\sum_{i=1}^{s} \alpha_i z(p_i)$, $\sum_{i=1}^{s} \alpha_i = 1$, is a stochastic vector, then for any word r, $r \in F_x$, the sum $\sum_{i=1}^{s} \alpha_i z(p_i r)$ is a stochastic vector.*

Another form of the representability criterion, which we give below, is analogous to the criterion for representability of events in finite deterministic automata proposed by M. O. Rabin and D. Scott (^{2}). For brevity of exposition and for clarity of the minimization process, we shall formulate the definitions and results in geometric terminology. We shall consider the countable-dimensional vector space E^∞ of sequences whose sums converge to unity. We shall denote by T^∞ the simplex whose point coordinates are nonnegative. Mappings K of the simplex T^∞ into itself will be called countably linear if they satisfy the condition

$$K\left(\sum_{i=1}^{\infty}\alpha_i v_i\right) = \sum_{i=1}^{\infty}\alpha_i K(v_i).$$

Every semigroup G of countably linear mappings of the simplex T^∞ into itself, having a generating set $K_{x_1}, K_{x_2}, \dots, K_{x_m}, \dots,\ x_i \in X$, induces in T^∞ a point multivalued mapping $G = \{z(p),\ p \in F_x\}$, if an initial point $z(e) \in T^\infty$ is specified by the condition $z(p) = K_p z(e)$. If a multivalued mapping $G = \{G, z(e)\}$ is fixed in T^∞, then the condition $(z(p) - z_0)\cdot \mathbf{n} > 0$, where $\mathbf{n}$ is a regular vector and $z_0 \in T^\infty$, defines some event $S,\ S \subset F_x$. With every event defined in the described way one can associate a certain equivalence relation in the space E^∞. A direction $\vec{\tau} = \mathbf{u} - \mathbf{v}$ in E^∞ will be called **internal** if there is a coordinate vertex $\mathbf{w}$ and a point $\mathbf{u}_1,\ \mathbf{u}_1 \in T^\infty$, such that

$$\mathbf{u} - \mathbf{v} = \lambda(\mathbf{w} - \mathbf{u}_1), \quad \lambda \neq 0, \quad \mathbf{w} \neq \mathbf{u}_1.$$

Let the predicate $H(\mathbf{u}, \mathbf{v}, \mathbf{R})$ be equal to one if and only if the vector $\mathbf{u}-\mathbf{v}$ either determines an internal direction or is a finite or countable linear combination of vectors defining internal directions of the R-equivalence, issuing from different vertices of the simplex. Define the equivalence

$$z_1 \equiv_G z_2 \sim (p) \quad \left(p \in F_x \to (K_p z_1)\cdot \mathbf{n} = (K_p z_2)\cdot \mathbf{n} \ \&\ H(K_p z_1, K_p z_2 \equiv_{G,\mathbf{n}})\right).$$

This equivalence is stable on the right with respect to transformations of the semigroup G, is countably linear, and the equivalence classes also form linear subspaces of E^∞. Thus, the quotient space $E^\infty/G, \mathbf{n}$ will be a linear space.

Theorem 2. In order that the event $S(G, \mathbf{n}, z_0)$ be representable in a finite probabilistic automaton, it is necessary and sufficient that the dimension of the quotient space $E^\infty/G, \mathbf{n}$, determined by the equivalence relation $\equiv_{G,\mathbf{n}}$, be finite. If the dimension of the quotient space $E^\infty/G, \mathbf{n}$ is finite, then it is equal to the minimal number of states of a probabilistic automaton representing the event $S(G, \mathbf{n}, z_0)$, determined by the given pair $G, \mathbf{n}$.

If the space E^N and the simplex T^N have already been determined by a finite probabilistic automaton representing the event S by the pair $G, \mathbf{n}$, then passing to the quotient space $E^N/G, \mathbf{n}$ determines a transition to an automaton with a smaller number of states, representing the same event S.

In conclusion we give an example of an event not representable in any finite probabilistic automaton. Let $\chi(N)$ be a predicate on the set of natural numbers whose verbal description is as follows: let $\bar{n}$ be the least integer solution of the inequality $n \cdot 2^{n+1} \geq N$, and let $i = N - (\bar{n} - 1)2^{\bar{n}}$. Let j be the least integer solution of the inequality $j(\bar{n}+1) \geq i$, and let $k = i - (j-1)(\bar{n}+1)$. Then

$$\chi(N) = \begin{cases} \alpha(j,k), & \text{if } k < \bar{n}+1, \\ 0, & \text{if } k = \bar{n}+1, \end{cases}$$

where $\alpha(j,k)$ is the value of the k-th digit of the number j, written in binary form. The function $\chi(N)$ is primitive recursive.

Remark. The event $T(x)$, defined by the primitive-recursive predicate $\chi(N)$ on the monogenic free semigroup F_x by the condition $p \in T(x) \sim \chi(|p|) = 1$, is not representable in any finite probabilistic automaton.

We note that if one defines a recursive event as an event for which there exists an algorithm $\mathfrak{A}$, defined for every word of the free semigroup F_x, that recognizes the event S and processes the words from S, and only those words, into a certain fixed letter of the alphabet $\mathfrak{F}$, then the event $T(x)$ turns out to be recursive. It remains unclear whether there exist events representable in finite probabilistic automata that would be nonrecursive.

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CITED LITERATURE

[1] M. O. Rabin, *Inform. and Control*, **6**, No. 3, 230 (1963); Russian transl. M. O. Rabin, *Kibernetich. sborn.*, issue 9, 1964, 121.
[2] M. O. Rabin, D. Scott, *Intern. Business Machines Corpor., J. Res. Develop.*, **3**, 114 (1959).

Note: Figure translations are in progress. See original paper for figures.

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