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Abstract

Full Text

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TOPOLOGICAL EQUIVALENCE OF CERTAIN CONES OF A BANACH SPACE

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Let us call, following M. G. Krein, a cone K of a Banach space E **normal** if there exists $\varepsilon > 0$ such that for any normalized elements x and y of the cone $\|x + y\| \geq \varepsilon$. Let us call a cone **strongly acute** if its elements satisfy the condition

$$\|x+z\| \geq \|x\| + \eta(x, \|z\|) \qquad (x, z \in K; \quad \eta(x, \|z\|) > 0 \text{ for } \|z\| > 0). \tag{1}$$

Theorem 1. *In a separable Banach space E with a normal cone K, one can introduce an equivalent norm with respect to which the cone K will be strongly acute.*

The proof follows immediately from the following two propositions.

Proposition 1. *In the space $C(S)$ of functions continuous on a compact set S, one can introduce an equivalent norm with respect to which the cone of nonnegative functions will be strongly acute.*

Proof. Let the diameter of the compact set S be taken to be equal to one. For each $\varepsilon > 0$ define a finite ε-covering, i.e., a collection of closed sets $S_i(\varepsilon) \subset S$ $(i = 1, 2, \dots, n(\varepsilon))$ with diameters not exceeding ε, covering S. For each ε define the functional

$$J(x, \varepsilon) = \frac{1}{n(\varepsilon)} \sum_{i=1}^{n(\varepsilon)} \max_{s \in S_i(\varepsilon)} |x(s)| \qquad (x(s) \in C(S);\ 0 < \varepsilon \leq 1) \tag{2}$$

and with its help define the desired norm

$$\|x\| = \sum_{k=0}^{\infty} 2^{-k} J(x, 2^{-k}). \tag{3}$$

It is easy to see that

$$\max_{s \in S} |x(s)| \leq \|x\| \leq 2 \max_{s \in S} |x(s)|.$$

Now consider nonnegative functions $x(s)$ and $z(s)$; $\|x\| = 1$, $\|z\| = \delta$. Choose $k_0 = k_0(x)$ so large that the oscillation of the function $x(s)$ on each of the sets $S_i(2^{-k_0})$ does not exceed $\frac{1}{2}\delta$. Then on that one of the sets $S_i(2^{-k_0})$ where $z(s)$ attains its greatest value,

$$\max[x(s) + z(s)] \geq \max x(s) + \delta/4$$

and therefore, according to (2),

$$J(x+z, 2^{-k_0}) \geq J(x, 2^{-k_0}) + \frac{1}{n(2^{-k_0})} \cdot \frac{\delta}{4}. \tag{4}$$

From (3) and (4) it follows that

$$\|x+z\| \geq \|x\| + \frac{2^{-k_0}}{n(2^{-k_0})} \cdot \frac{\delta}{4},$$

which proves Proposition 1.

Proposition 2. *A separable Banach space with a normal cone K can be embedded isomorphically in the space $C(S)$, and K (and only K) is mapped into the cone of nonnegative functions.*

This proposition, generalizing the Banach–Mazur theorem on a universal space, was proved by M. G. Krein ([1]).

We now define in the separable Banach space E a complete minimal system

$$x_1, x_2, x_3, \ldots \qquad (\|x_i\| = 1) \tag{5}$$

with total adjoint $\{f_i\}_1^\infty$. Consider the smallest closed cone K spanned by the elements of the system (5). Suppose that it is strongly acute.

Lemma 1. *If $x \in K$, then $\lim_{n\to\infty} S_n x = x$, where*

$$S_n x = \sum_{k=1}^{n} f_k(x) x_k.$$

The proof rests on the monotonicity of the norm in K.

Consider the set D_0 of those elements $x \in K$ for which $\|x\| \leq 1$. For each $y \in D_0$ define the set

$$D(y) = \{z : z - y \in K,\ \|z\| \leq 1\}.$$

Lemma 2. *If $y_k \to y$ $(y_k \in D_0)$, then*

$$\lim_{k\to\infty} \rho(D(y_k), D(y)) = 0 \ * .$$

Proof. The case $\|y\| < 1$ is examined quite simply without using any special properties of the cone K. Let $\|y\| = 1$. Since the cone K is strongly acute, $D(y)$ contains only the point y. Let z_k be an arbitrary point of the set $D(y_k)$. Then

$$\lim_{k\to\infty} \|y - y_k + z_k\| \le \lim_{k\to\infty} \|y - y_k\| + \lim_{k\to\infty} \|z\| = 1,$$

and, according to condition (1), $\lim \|y_k - z_k\| = 0$, i.e. the diameter of the set $D(y_k)$ tends to zero as $k \to \infty$. Since at the same time the point y_k itself tends to y, the distance between $D(y_k)$ and $D(y)$ tends to zero.

For each normalized element $x \in D_0$, define the sequence of numbers

$$\delta_n(x) = \frac{d\{D(S_n x)\}}{d\{D_0\}} \prod_{k=1}^{n} \left(1 - \frac{f_k(x)}{2^k \|f_k\|}\right) \qquad (n = 1, 2, ...),$$

where $d\{M\}$ is the diameter of the set M. From Lemmas 1–2 it follows that

$$1 \ge \delta_1(x) \ge \delta_2(x) \ge \cdots; \qquad \lim_{n\to\infty} \delta_n(x) = 0.$$

Lemma 3. *Let the sequence of numbers Δ_i be subject to the conditions*

$$1 \ge \Delta_1 \ge \Delta_2 \ge \cdots; \qquad \lim_{n\to\infty} \Delta_n = 0.$$

Then there exists a unique normalized element $x \in D_0$ for which

$$\delta_n(x) = \Delta_n \qquad (n = 1, 2, ...).$$

* $\rho(X, Y)$ is the Hausdorff distance between subsets of a metric space.

Proof. Consider the system of equations

$$d\left\{D\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\} \prod_{i=1}^{n} \left(1 - \frac{\lambda_i}{2^i \|f_i\|}\right) = d\{D_0\}\Delta_n \quad (n = 1, 2, ...). \qquad (6)$$

For fixed values $\lambda_i \ge 0$ ($i < n$), the left-hand side of the equation is a strictly decreasing function of $\lambda_n \ge 0$, vanishing when

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| = 1,$$

and taking the value $d\{D_0\}\Delta_{n-1}$ when $\lambda_n = 0$. From what has been said it follows that the system (6) has a unique solution $\{\lambda_i\}_1^n$ $(\lambda_n \geq 0)$. The sets

$$D\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

form a decreasing sequence, and their diameters decrease without bound as $n \to \infty$. The unique element

$$x = \sum_{i=1}^{\infty} \lambda_i x_i,$$

lying in their intersection, will be the desired element.

Theorem 2. *A normal cone K, spanned by the vectors of a complete minimal system with total adjoint, is homeomorphic to the cone $K(l_2)$, spanned by the orthonormal basis of the space l_2.*

Proof. By Theorem 1, without loss of generality one may assume that K is strongly acute. With the aid of the constructions carried out above, to each element $x \in K$ we assign, one-to-one, the nonincreasing sequence

$$\Delta_n = \delta_n\left(\frac{x}{\|x\|}\right)\|x\| \quad (n = 0, 1, 2, \ldots;\ \Delta_0 = \|x\|), \tag{7}$$

which tends to zero as $n \to \infty$. The resulting correspondence is also mutually continuous if on the set $\widetilde{K}$ of sequences (7) we introduce the topology of coordinatewise convergence. Let us show this. Suppose the sequence $y_\nu \in K$ converges to y. Without loss of generality one may assume $\|y_\nu\| = 1$. From the convergence of y_ν to y it follows that $f_n(y_\nu) \to f_n(y)$ and $S_n y_\nu \to S_n y$ $(n = 1, 2, \ldots)$, whence, by Lemma 2,

$$\delta_n(y_\nu) \to \delta_n(y) \quad (n = 1, 2, \ldots). \tag{8}$$

On the other hand, if (8) holds, then, considering (6) successively for $n = 1, 2, \ldots$, we conclude that

$$\lim_{\nu\to\infty} S_n y_\nu = S_n y \quad (n = 1, 2, \ldots).$$

Given an arbitrary $\varepsilon > 0$, choose n so large that $d\{D(S_n y)\} < \varepsilon/2$. Then choose ν so that

$$\rho(D(S_n y); D(S_n y_\nu)) < \varepsilon/2,$$

which can be done on the basis of Lemma 2. Since $y_\nu \in D(S_n y_\nu)$, and $y \in D(S_n y)$, it follows from the preceding inequalities that $\|y_\nu - y\| < \varepsilon$. Thus, every cone K satisfying the conditions of Theorem 2 is homeomorphic to $\widetilde{K}$. Since the cone $K(l_2)$ satisfies the conditions of Theorem 2, K is homeomorphic to $K(l_2)$.

Theorem 3. *Let E be a Banach space with an unconditional basis.* * *Then the cone spanned by the elements of this basis is homeomorphic to $K(l_2)$.*

The proof, according to Theorem 2, follows from the fact that the cone spanned by the elements of an unconditional basis is normal.

Theorem 4. *Every Banach space with an unconditional basis is homeomorphic to l_2.*

Proof. We first establish a topological correspondence between the cone K, spanned by the elements of the unconditional basis

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* For the definition and properties of an unconditional basis see ([3]).

$\{x_k\}_1^\infty$ of the space E, and $K(l_2)$, generated by the orthogonal basis $\{e_k\}_1^\infty$ of the space l_2. Let $x = \sum_1^\infty a_k x_k \in E$. If to the element $x' = \sum a'_k |x_k| \in K$ there corresponds the element $y' = \sum b_k e_k \in K(l_2)$, then to the element x we assign the element $y = \sum b_k \operatorname{sign} a_k \cdot e_k$. This will be the required homeomorphism.

The result of Theorem 4 was established in another way by Cz. Bessaga and A. Pełczyński ([2]); in their proof a special case of Theorem 4 is used (a homeomorphism of the spaces C and l), obtained earlier by the author ([4]). H. Corson and V. Klee ([5]) established a homeomorphism between l_2 and $K(l_2)$; from this result and Theorems 3 and 4 one obtains:

Theorem 5. *A space E with an unconditional basis is homeomorphic to its cone $K(E)$.*

It is unknown whether in Theorems 3–5 one can dispense with the requirement that the basis be unconditional.

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Note: Figure translations are in progress. See original paper for figures.