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APPLICATION OF CONTINUAL INTEGRALS TO THE DERIVATION OF SHORT-WAVE ASYMPTOTICS IN DIFFRACTION PROBLEMS

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The short-wave asymptotics in various diffraction problems that do not admit an exact solution usually cannot be obtained systematically. It is often necessary to use certain additional considerations of a physical nature. It seems to us that this can be avoided by using a representation of the solutions by means of continual integrals and applying the steepest-descent method to these integrals. In the present note this assertion is illustrated by the example of the stationary problem of diffraction by smooth convex bodies. Almost all formulas obtained in this work had already been derived earlier for the two-dimensional and three-dimensional cases by other methods; however, the method of continual integrals has the advantage of simplicity and universality.

Let d be an infinite domain of n-dimensional Euclidean space, having a smooth boundary l, and such that its complement is convex. Consider the following problem:

$$\frac{\partial G_\mp}{\partial t} = \Delta_x G_\mp, \qquad G_\mp = G_\mp(x, x'; t) \quad (x, x' \in d), \tag{1}$$

$$G_-\Big|_{x\in l} = 0, \qquad \frac{\partial G_+}{\partial n}\Big|_{x\in l} = 0, \qquad G_\mp \xrightarrow[t\to 0]{} \delta(x - x').$$

The Green' s function $\Gamma_\mp(x, x'; k)$ of the stationary diffraction problem $(-\Delta_x \Gamma_\mp - k^2\Gamma_\mp = \delta(x - x'))$ is expressed in terms of $G_\mp(x, x'; t)$ by the integral

$$\Gamma_\mp(x, x'; t) = \int_C dt\, e^{k^2 t} G_\mp(x, x'; t), \tag{2}$$

in which C is some contour in the complex t-plane going from the point $t = 0$ to an infinitely distant point. In the present work we derive the asymptotics of the function G as $t \to 0$. From it, by means of formula (2), the short-wave ($k \to \infty$) asymptotics of Γ is then easily obtained. The behavior of the function G as $t \to 0$ apparently is also of independent interest.

Let us write the function G with the aid of a continual integral. Introduce a "two-sheeted" domain D, consisting of two copies of the domain d glued together along the boundary l. We mark one of the copies with the sign $+$, the other with the sign $-$. Points of D will be denoted by capital letters X. The notation $X = x_+$ ($X = x_-$) means that X coincides with the point $x \in d$ lying on the $+$ ($-$) sheet. Consider the continual integral, which is symbolically written in the following form

$$Q(X, X'; t) \equiv \int_{C(D)} \exp\left\{-\frac{1}{4}\int_0^t d\tau \left(\frac{d\overline{X}}{d\tau}\right)^2\right\} \prod_{\tau=0}^{t} dS(\tau); \tag{3}$$

integration here is carried out over the space $C(D)$ of continuous curves $\bar{X}(\tau)$ ($0 \leq \tau \leq t$), lying in D and satisfying the condition $\bar{X}(0) = X'$, $\bar{X}(t) = X$. Integral (3) may be regarded as the limit of the finite-multiple integrals

$$\int_D \cdots \int_D \exp\left\{-\frac{1}{4}\sum_{k=0}^{m}\frac{(\bar{X}_{k+1}-\bar{X}_k)^2}{\tau_{k+1}-\tau_k}\right\} \frac{1}{[4\pi(\tau_1-\tau_0)]^{n/2}} \prod_{k=1}^{m} \frac{dS_k}{[4\pi(\tau_{k+1}-\tau_k)]^{n/2}}; \tag{4}$$

dS_k is an element of n-dimensional volume, $\bar{X}_0 = X'$, $\bar{X}_{m+1} = X$, and the limit is taken in the direction formed by the partitions $q_m(\tau_0, \tau_1, \dots, \tau_{m+1})$ of the interval $[0, t]$. The function G is expressed in terms of integral (3) by the formula

$$G_\pm(x, x'; t) = Q(x_+, x'_+; t) \mp Q(x_+, x'_-; t). \tag{5}$$

The asymptotics of the function Q as $t \to 0$ can be found by the method of steepest descent. The large parameter is the quantity t^{-1}. The saddle points are determined from the condition that the functional

$$\int_0^t d\tau \left(\frac{d\bar{X}}{d\tau}\right)^2 \qquad (\bar{X} \in D,\ \bar{X}(0) = X',\ \bar{X}(t) = X) \tag{6}$$

be minimized.

We shall assume the domain d to be such that this variational problem has a unique solution.* The solution is called: a) a direct ray, if it is a straight-line segment; b) a reflected ray, if it is a two-link broken line; c) a creeping ray, if it is a curve having more than one common point with the boundary l. In the last case an entire arc of the creeping ray lies on l. These three possibilities exhaust all cases.

Direct and reflected rays. Suppose that the direct and reflected rays under consideration do not touch the boundary l. The asymptotics of the function Q as $t \to 0$ is determined by integration over a small neighborhood of the ray. In this neighborhood introduce ray coordinates Φ and α: Φ is the shortest distance in the domain D from X' to the variable point $\bar{X}$, and $\alpha = \{\alpha^1, \dots, \alpha^{n-1}\}$ is some parametrization of the family of lines realizing this shortest distance. The ray under consideration belongs to the indicated family, and let on it $\alpha = \alpha_0$. Making in integral (3) a change of variables, we obtain

$$Q \sim \int \exp\left\{-\frac{1}{4}\int_0^t d\tau \left[\left(\frac{d\Phi}{d\tau}\right)^2 + \sum A_{ij}\frac{d\alpha^i}{d\tau}\frac{d\alpha^j}{d\tau}\right]\right\} \times$$

$$\times \prod_{\tau=0}^{t} I(\Phi,\alpha)\, d\Phi(\tau) \prod_{i=1}^{n-1} d\alpha^i(\tau). \qquad (7)$$

We shall not write out the expression for $A_{ij} = A_{ij}(\Phi,\alpha)$, while $I(\Phi,\alpha) = D(x)/D(\Phi,\alpha)$. Since, in the integration, the points $\{\Phi(\tau),\alpha(\tau)\}$ ($\Phi \geq 0$, $\Phi(0) = 0$, $\Phi(t) = \Phi_R$ —the length of the ray, $\alpha(0) = \alpha(t) = \alpha_0$) lie in a small neighborhood of the ray $\{\Phi = \Phi_R(\tau/t),\ \alpha = \alpha_0\}$, which gives the functional (6) a minimum, we put in integral (7) $A_{ij}(\Phi,\alpha) \to A_{ij}(\Phi_R(\tau/t),\alpha_0) \equiv A_{ij}(\tau)$ and $I(\Phi,\alpha) \to I(\Phi_R(\tau/t),\alpha_0) \equiv I(\tau)$. After this we arrive at a Gaussian integral, which is evaluated explicitly. Note that as $\tau \to 0$, $I(\tau) \sim I_0(\alpha_0)(4\pi\tau)^{n-1}$. By a choice of the variables α one can make $I_0(\alpha_0) = 1$. Then the asymptotics turns out to be the following:

$$Q \sim [4\pi t\, I(t)]^{-1/2} \exp\left\{-\frac{1}{4}\frac{\Phi_R^2}{t}\right\}. \qquad (8)$$

* For $n = 2$ it is sufficient for this that the contour l be infinite. If the contour is finite, then one should pass to an infinite one, using in diffraction a device based on the introduction of a certain "screw" surface.

The quantity $I(t)$ characterizes the geometrical spreading of the rays at the point X. For the direct ray $I(t) = (4\pi tt)^{n-1}$.

The creeping ray. We shall restrict ourselves here to the case $X, X' \in l$. Then the creeping ray coincides with the geodesic on l joining X and X'. Through the point X', orthogonally to this geodesic, draw on l a smooth surface of dimension $n-2$ and construct a field of geodesics orthogonal to it, covering a neighborhood of the ray. In this neighborhood on l introduce the variables σ and β: σ is the field function of the corresponding variational problem, and $\beta = \{\beta^1, \dots, \beta^{n-2}\}$ is some parametrization of the field. The ray under consideration belongs to the field, and let on it $\beta = \beta_0$. We shall describe a current point $\bar{X} \in D$ by the variables n, σ, β: n is the distance along the normal from $\bar{X}$ to l, if $\bar{X} = x_+$, and the same distance with a minus sign if $\bar{X} = x_-$; σ and β are the coordinates of the foot of the normal. Introducing new variables into integral (3), we obtain

$$Q \sim \int \exp\left\{-\frac{1}{4}\int_0^t d\tau \left[\left(\frac{dn}{d\tau}\right)^2 + \left(1+\frac{|n|}{\rho}\right)^2 \left(\frac{d\sigma}{d\tau}\right)^2 + \sum B_{ij}\frac{d\beta^i}{d\tau}\frac{d\beta^j}{d\tau}\right]\right\} \times$$

$$\times \prod_{\tau=0}^{t}\left(1+\frac{|n|}{\rho}\right) I(n,\sigma,\beta)\, dn(\tau)\, d\sigma(\tau) \prod_{i=1}^{n-2} d\beta^i(\tau). \qquad (9)$$

Here $\rho = \rho(\sigma, \beta)$ is the radius of curvature of the geodesic at the point σ, β; $\mathcal{B}_{ij} = \mathcal{B}_{ij}(n, \sigma, \beta)$ and

$$\left(1+\frac{|n|}{\rho}\right) I(n,\sigma,\beta) \equiv \frac{D(x)}{D(n,\sigma,\beta)}.$$

When the points $\{n(\tau), \sigma(\tau), \beta(\tau)\}$ are integrated, $(n(0) = n(t) = 0,\ \sigma(0) = 0,\ \sigma(t) = \Phi_s$ is the length of the ray, $\beta(0) = \beta(t) = \beta_0)$, they lie in a small neighborhood of the ray $\{n = 0,\ \sigma = \sigma_0(\tau) \equiv \Phi_s \frac{\tau}{t},\ \beta = \beta_0\}$; therefore, asymptotically one may put

$$Q \sim \int \exp\left\{-\frac{1}{4}\int_0^t d\tau \left[\left(\frac{dn}{d\tau}\right)^2 + 2\frac{|n|}{\rho(\tau)}\left(\frac{d\sigma_0}{d\tau}\right)^2 + \left(\frac{d\zeta}{d\tau}\right)^2 + \sum B_{ij}(\tau)\frac{d\beta^i}{d\tau}\frac{d\beta^j}{d\tau}\right]\right\} \prod_{\tau=0}^{t} I(\tau)\, dn(\tau)\, d\zeta(\tau) \prod_{i=1}^{n-2} d\beta \qquad (10)$$

The notation is analogous to that introduced earlier. Integral (10) is the product of two Gaussian integrals in ζ and β and the integral

$$\psi \equiv \int \exp\left\{-\frac{1}{4}\int_0^t d\tau \left[\left(\frac{dn}{d\tau}\right)^2 + 2\frac{|n|}{\rho(\tau)}\left(\frac{d\sigma_0}{d\tau}\right)^2\right]\right\} \prod_{\tau=0}^{t} dn(\tau). \qquad (11)$$

Using the connection of integral (11) with a parabolic differential equation (Kac' s formula), it is not difficult to find its asymptotics as $t \to 0$. It turns out to be the following:

$$\psi \sim \psi_1 \equiv A^{1/3}(0) A^{1/3}(t) \left(-\frac{1}{2\pi i}\right) \int_{C_1} d\zeta\, e^{-T\zeta} \frac{v(-\zeta)}{2v'(-\zeta)}. \qquad (12)$$

Here $v(\zeta)$ is the solution, exponentially decreasing as $\zeta \to +\infty$, of the equation $v'' = \zeta v$; the contour C_1 encircles, in the positive direction, the half-axis $\zeta \geqslant 0$,

$$A^2(\tau) = 2\left(\frac{\Phi_s}{2t}\right)^2 \frac{1}{\rho(\tau)} \quad \text{and} \quad T = \int_0^t A^{4/3}(\tau)\, d\tau.$$

The final formula has the form

$$Q \sim [4\pi t I(t)]^{-1/2} \psi_1 \exp\left\{-\frac{1}{4}\frac{\Phi_s^2}{2}\right\}. \tag{13}$$

The Jacobian $I(t)$ is normalized by the condition $I(t) \sim (4\pi t)^{n-2}$.

With formulas (8) and (13) for the function $Q^{\tau\to 0}$ we shall conclude this note. The scheme described for obtaining them is very simple; however, we cannot rigorously justify all the transformations over continual integrals. Nevertheless, in the two-dimensional case, for $|\arg t| < \pi/2$, the final formulas can be justified by means of the techniques proposed for this purpose in diffraction problems (see, for example, [1]). For $|\arg t| = \pi/2$, these techniques make it possible to obtain a justification only for formulas corresponding to the direct and reflected rays. Of known interest is the consideration of the case $|\arg t| > \pi/2$. Here the derivation of the asymptotics requires an additional "deformation of the contour" in the continual integral (3).

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[1] V. S. Buslaev, DAN, **145**, No. 4, 753 (1962).

Note: Figure translations are in progress. See original paper for figures.