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Abstract

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ABSOLUTE STABILITY OF AUTOMATIC SYSTEMS WITH ONE PULSE CONTROLLER

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1°. In the present paper the method of matrix inequalities of V. A. Yakubovich ([1-5]) is extended to the case of difference equations of the form

$$x_{t+1} = Px_t + q\varphi(\sigma_t), \qquad \sigma_t = r^* x_t, \qquad t = 0, 1, 2, \dots, \tag{1}$$

which describe pulse control systems. Here x, q, r are $\nu \times 1$-vectors; P is a nonsingular $\nu \times \nu$-matrix whose spectrum lies inside the unit circle*; $\varphi(\sigma)$ is a differentiable function satisfying the conditions:

$$\text{a)}\ \ \varphi(0) = 0, \qquad 0 \le \sigma\varphi(\sigma) \le \mu_0\sigma^2; \qquad \text{b)}\ \ \lim_{|\sigma|\to\infty} |\sigma|^{-2} \left| \int_0^\sigma \varphi\, d\sigma - \frac{1}{2}\varphi\sigma \right| = 0; \tag{2}$$

$$a_1 \le \varphi'(\sigma) \le a_2, \qquad a_1 \le 0, \qquad a_2 \ge \mu_0 > 0. \tag{3}$$

For the first time for systems (1), a frequency condition for absolute stability in the class of nonlinearities satisfying relation (2a) was obtained by Ya. Z. Tsypkin ([6]). Under assumptions (2a), (3), R. Kalman and G. Szegő obtained ([7-9]) new criteria. Below, under some additional inessential assumptions, criteria of absolute stability will be obtained that are broader than conditions ([6-9]).

Introduce the notation

$$\chi(\lambda) = r^*(P - \lambda I)^{-1} q, \qquad \chi_0 = \chi(0), \qquad \beta_j = \alpha_j(\alpha_j\chi + 2), \qquad j = 1, 2;$$

$$\Phi(\lambda, \vartheta, \xi) = \mu_0^{-1} + \operatorname{Re}\chi(\lambda) - \vartheta\left[2\operatorname{Re}(1-\lambda)\chi(\lambda) + \alpha_0|(1-\lambda)\chi(\lambda)|^2\right] + \xi|1-\lambda|^2\left[\chi_0 - 2\operatorname{Re}\chi(\lambda) - \beta_0|\chi(\lambda)|^2\right], \tag{4}$$

where the quantity a_0 may be assigned the values $a_0 = a_1$ or $a_0 = a_2$, while the choice of β_0 is restricted by the conditions: $\beta_0 = \beta_1$, if $(\beta_2 - \beta_1)(\chi_0 + a_1^{-1}) \leq 0$, $\beta_0 = \beta_2$, if $(\beta_2 - \beta_1)(\chi_0 + a_2^{-1}) \geq 0$**.

The main result of the paper can be formulated as follows:

Theorem. *If for system (1) there exist real numbers ϑ, ξ satisfying the inequalities: $\vartheta \leq 0$ for $a_0 = a_1$, $\vartheta \geq 0$ for $a_0 = a_2$, $0 \leq \xi(\chi_0 + a_1^{-1})^{-1}$ for $\beta_0 = \beta_1$, $0 \leq \xi(\chi_0 + a_2^{-1})^{-1}$ for $\beta_0 = \beta_2$, and such that $\Phi(\lambda, \vartheta, \xi) > 0$ for all λ, $|\lambda| = 1$, then the trivial solution of this system is stable in the large for any differentiable functions $\varphi(\sigma)$ satisfying conditions (2), (3).*

For a given frequency response $\chi(e^{i\omega})$, $0 \leq \omega \leq 2\sigma$, the question of the existence of numbers ϑ, ξ satisfying the condition of the theorem is easily solved geometrically, similarly to how this was done in paper ([5]).

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* All matrices and vectors here and below are real, unless complex numbers explicitly enter into their expressions. The asterisk denotes transposition (in the case of complex expressions also conjugation).

** It is easy to check that in the case when $a_2^{-1} \geq -\chi_0 \geq a_1^{-1}$, both values $\beta_0 = \beta_1, \beta_2$ are possible.

Let us note three special cases that simplify the conditions of the theorem:

1) If in relations (3) $\alpha_2 = -\alpha_1 = \alpha_0 > 0$, then the expression $\Phi(\lambda, \vartheta, \xi)$ has the form

$$\Phi(\lambda, \vartheta, \xi) = \mu_0^{-1} + \xi\chi_0|1 - \lambda|^2 + \mathrm{Re}\{[1 - 2\vartheta(1 - \lambda) -$$

$$-2\xi|1 - \lambda|^2]\chi(\lambda)\} - \alpha_0(|\vartheta| + 2|\xi| + \xi\alpha_0\chi_0)|(1 - \lambda)\chi(\lambda)|^2,$$

where ϑ, ξ are some arbitrary fixed real numbers, the choice of the sign of ξ being restricted by the conditions: $\xi \leq 0$ for $\chi_0 \leq -\alpha_0^{-1}$, $\xi \geq 0$ for $\chi_0 \geq \alpha_0^{-1}$, and ξ arbitrary for $-\alpha_0^{-1} < \chi_0 < \alpha_0^{-1}$.

We note that in this case, and for $\xi = 0$, the conditions of the theorem coincide exactly with the Segé criterion ([7]), if in the latter the boundary of the stability region is excluded.

2) If in relations (3) $\alpha_1 = 0$ and $\chi_0 \geq -2\alpha_2^{-1}$, then, restricting ourselves to the choice $\alpha_0 = \alpha_1 = 0$, $\beta_0 = \beta_1 = 0$, the conditions of the theorem may be written in the form

$$\mu_0^{-1} - \frac{1}{2}\xi\chi_0|1 - \lambda|^2 + \mathrm{Re}\{[1 + \vartheta_1(1 - \lambda) + \xi_1|1 - \lambda|^2]\chi(\lambda)\} > 0$$

for all λ, $|\lambda| = 1$, and for some fixed positive numbers ξ_1, ϑ_1.

3) If in relations (3) $\alpha_1 = -\infty,\ \alpha_2 = +\infty$, then it is necessary that $\xi = \vartheta = 0$, and the conditions of the theorem coincide with the criteria of Tsypkin ([6]) and Segé-Kalman ([8,9]).

2°. Lemma 1. *Suppose that for the system*

$$x_{t+1} = f(x_t), \qquad f(0) = 0, \qquad t = 0, 1, 2, \dots, \tag{5}$$

where $f(x)$ is a vector function continuous in the whole space $\{x\}$, there exists a continuous scalar function $V(x)$ having the following properties:

I. $V(0) = 0$. II. $\lim_{|x|\to\infty} V(x) = \infty$. III. $\Delta V(x) \equiv V[f(x)] - V(x) \le 0$ for all x. IV. From $\Delta V(x_t) \equiv 0$ it follows that $x_t \equiv 0$, if x_t is a solution of equations (5). *Then $V(x) > 0$ for all $x \ne 0$, and the trivial solution of system (5) is stable in the large in the sense of Lyapunov, i.e., $x_t \to 0$ as $t \to \infty$, and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_t| < \varepsilon$ for $|x_0| < \delta$ and for all $t \ge 0$.*

The proof of Lemma 1 is carried out according to scheme (5). By III and (5) we have $V(x_t) \le V(x_0)$; therefore, from II it follows that $|x_t| \le \text{const}$. Let y be an arbitrary limit point of the sequence x_t. There exists a subsequence t' such that $\lim_{t'\to\infty} x_{t'} = y$, and, by virtue of the continuity of $V(x)$,

$$\lim_{t'\to\infty} V(x_{t'}) = V(y).$$

Since the sequence $V(x_t)$ is monotone, it follows that

$$\lim_{t\to\infty} V(x_t) = V(y).$$

Obviously, $x_t = x(t, x_0)$ is a continuous function of the initial values x_0. Therefore

$$\lim_{t'\to\infty} x_{t'+\tau} = \lim_{t'\to\infty} x(\tau, x_{t'}) = x(\tau, y),$$

$$V(y) = \lim_{t'\to\infty} V[x_{t'+\tau}] = V[x(\tau, y)],$$

whence we have $\Delta V[x(\tau, y)] \equiv 0,\ \tau = 0, 1, \dots$. Consequently, by property IV, $x(\tau, y) \equiv 0$. In particular, $x(0, y) = y = 0$, and, since y is an arbitrary limit point, $x_t \to 0$ as $t \to \infty$.

Suppose that for some vector z one has $V(z) \le 0$. Setting in (5) $x_0 = z$, we have, according to III,

$$V[x(t+\tau, z)] \le V[x(t, z)] \le V(z) \le 0, \qquad t = 0, 1, \dots, \quad \tau = 0, 1, \dots$$

Passing to the limit as $\tau \to \infty$, by what has been proved we obtain

$$\lim_{\tau\to\infty} x(t+\tau, z) = 0$$

and, by virtue of I,

$$\lim_{t\to\infty} V[x(t+\tau, z)] = 0.$$

Therefore $0 \leq V[x(t,z)] \leq 0,\ t = 0, 1, ...,$ i.e. $V[x(t,z)] \equiv 0$. By property IV, we have $x(t,z) \equiv 0$; in particular, $x(0,z) = z = 0$. Thus, for all $z \neq 0$ one has $V(z) > 0$.

Let $\varepsilon > 0$ be given. By what has been proved,

$$V_\varepsilon = \mathrm{Inf}_{|y|\geq\varepsilon} V(z) > 0.$$

By virtue of properties I, III and the continuity of the function $V(x)$, there exists $\delta > 0$ such that $V(x_t) \leq V(x_0) < V_\varepsilon$ in the case $|x| < \delta$. If for some t at

if for $|x_0| < \delta$ the inequality $|x_t| \geqslant \varepsilon$ holds, then $V(x_t) \geqslant V_\varepsilon$. A contradiction shows that for $|x_0| < \delta$, for all t one has $|x_t| < \varepsilon$.

Lemma 2. Let $H = H^*$ and let P be some $v \times v$ matrix, with the spectrum of P lying inside the unit circle; q, r_1 are $v \times 1$ vectors; γ_0, φ are real numbers. Denote

$$\Psi(x,\varphi) = x^*Gx + 2g^*x\varphi + \gamma\varphi^2,$$

where $G = H - P^*HP$, $-g = \frac{1}{2}r_1 + P^*Hq$, $\gamma = \gamma_0 - q^*Hq$. For the existence of a matrix $H = H^* > 0$ such that $\Psi(x,\varphi) > 0$ for any x, φ, $|x| + |\varphi| \neq 0$, it is necessary and sufficient that

$$\Phi_0(\lambda) \equiv \gamma_0 + \mathrm{Re}\, r_1^*(P - \lambda I)^{-1}q > 0$$

for all $\lambda,\ |\lambda| = 1$.

Proof. Introduce the matrix $A = (P - I)(P + I)^{-1}$, which, by the properties of the matrix P, is Hurwitz, and perform the nonsingular transformation

$$x = \sqrt{2}(P + I)^{-1}(x_1 - q\varphi_1), \qquad \varphi = \sqrt{2}\varphi_1.$$

Then

$$\Psi(x,\varphi) = \Psi_1(x_1,\varphi_1) = x_1^*G_1x_1 + 2g_1^*x_1\varphi_1 + \gamma_1\varphi_1^2,$$

where

$$-G_1 = A^*H + HA, \qquad -g_1 = Ha + b,$$

$$\gamma_1 = 2[\gamma_0 + r_1^*(P + I)^{-1}q], \qquad a = -2(P + I)^{-1}q, \qquad b = -(P^* + I)^{-1}r_1,$$

and the condition $\Phi_0(\lambda) > 0$ for all $\lambda,\ |\lambda| = 1$, is equivalent to the inequality

$$\Phi_1(i\omega) = \gamma_1 + 2\operatorname{Re} b^*(A - i\omega I)^{-1} a > 0,$$

satisfied for all $\omega \geqslant 0$, which, by theorem 1 ([1]), is necessary and sufficient for the existence of a matrix $H > 0$ satisfying the relation

$$\gamma_1 G_1 - g_1 g_1^* > 0,$$

which, together with the inequality $\gamma_1 = 2\Phi_0(-1) > 0$, proves Lemma 2.

Lemma 3. If the spectrum of the matrix P lies inside the unit circle and

$$G = H - P^* H P > 0,$$

then also $H > 0$.

The proof follows from the formula

$$H = \sum_{t=0}^{\infty} P^{*t} G P^t,$$

which gives the unique, as is readily checked, solution of the equation $G = H - P^* H P$.

Proof of the theorem. Suppose that the conditions of the theorem are satisfied. Consider the function

$$V(x) = x^* H_1 x + 2h^* x \varphi(\sigma) + \varkappa \varphi(\sigma)^2 + 2\vartheta \int_0^{\sigma} \varphi(\zeta)\, d\zeta, \qquad \sigma = r^* x, \quad (6)$$

where $h = \xi (I - P^{-1})^* r$, $\varkappa = \xi \chi_0$, $H_1 = H + H_0$; H_0 is determined from the equation

$$H_0 - P^* H_0 P = R \equiv (\vartheta \alpha_0 + \xi \beta_0)(I - P^*) r r^* (I - P);$$

$\chi_0, \alpha_0, \beta_0$ are the same as in (4), and the parameters ξ, ϑ are to be chosen in accordance with the conditions of the theorem.

It is obvious that the function (6) has property I of Lemma 1. Denoting $\sigma_{t+1} - \sigma_t = \Delta\sigma$, $\varphi(\sigma_t) = \varphi_t$, $\varphi_{t+1} - \varphi_t = \Delta\varphi$, we obtain from (6), (1), (4)

$$-\Delta V(x_t) = \Psi(x_t, \varphi_t) + \Omega,$$

where

$$\Omega = (\sigma_t \varphi_t - \mu_0^{-1} \varphi_t^2) + 2\vartheta \left[\varphi_t \Delta\sigma + \frac{1}{2} \alpha_0 \Delta\sigma^2 - \int_{\sigma_t}^{\sigma_{t+1}} \varphi \, d\sigma \right] - \xi \left[\chi_0 \left(\frac{\Delta\varphi}{\Delta\sigma} \right)^2 + 2 \frac{\Delta\varphi}{\Delta\sigma} - \beta_0 \right] \Delta\sigma^2 \geqslant 0$$

by virtue of the conditions on $\varphi(\sigma)$, $\alpha_0, \beta_0, \xi, \vartheta$, and $\Psi(x, \varphi)$ has the same form as in Lemma 2, if r_1 and γ_0 are defined by the relations

$$\begin{aligned} r_1 = 2P^* H_0 q &+ [1 - 2\vartheta(1 - \alpha_0 r^* q) - 2\xi(2 + \beta_0 r^* q)]\, r \\ &+ 2\,[\vartheta(1 + \alpha_0 r^* q) + \xi(1 + \beta_0 r^* q)]\, P^* r + 2\xi P^{*-1} r, \end{aligned} \quad (7)$$

$$\gamma_0 = -q^* H_0 q + \mu_0^{-1} - \vartheta r^* q(2 + \alpha_0 r^* q) + \xi \left[2\chi_0 - 2r^* q - \beta_0 (r^* q)^2\right].$$

For convenience in estimating, in the expression $-\Delta V(x_t)$, obtained from (6), (1), the terms

$$(\sigma_t \varphi_t - \mu_0^{-1} \varphi_t^2) + 2\vartheta(\varphi_t \Delta\sigma + \alpha_0 \Delta\sigma^2/2) + \xi\beta_0 \Delta\sigma^2$$

have been added and subtracted.

From the equation for H_0 it follows that

$$2 \operatorname{Re} q^* H_0 P (P - \lambda I)^{-1} q - q^* H_0 q = -q^* (P^* - \lambda^* I)^{-1} R (P - \lambda I)^{-1} q$$

for $|\lambda| = 1$. With the help of this identity and the relations (7), (4), it is easy to verify that the expressions for $\Phi(\lambda, \vartheta, \xi)$ of the theorem and $\Phi_0(\lambda)$ of Lemma 2 coincide. Therefore, in the conditions of the theo-

therefore, by Lemma 2, there exists a matrix $H = H^* > 0$ for which $\Psi(x, \varphi) > 0$ for all $x, \varphi,\ |x| + |\varphi| \neq 0$. Substituting this matrix into expression (6), we obtain that function (6) has properties III, IV of Lemma 1 and, moreover, $-\Delta V(x) > 0$ for all $x \neq 0$.

Let us now show that function (6) also satisfies condition II of Lemma 1. Denote

$$P_\mu = P + \mu q r^*, \qquad H_1(\mu) = H_1 + \mu(h r^* + r h^*) + \mu(\mu \varkappa + \vartheta) r r^*, \qquad 0 \le \mu \le \mu_0. \tag{8}$$

For the special case $\varphi(\sigma) = \mu\sigma$, from (1), (6), (8) we have $x_{t+1} = P_\mu x_t$, $V(x) = x^* H_1(\mu) x$, $-\Delta V(x) = x^*[H_1(\mu) - P_\mu^* H_1(\mu) P_\mu] x$. According to what has been set forth, $-\Delta V(x) > 0$ for all $x \neq 0$, i.e. $H_1(\mu) - P_\mu^* H_1(\mu) P_\mu > 0$. It is easy to prove (by contradiction) that, under the conditions of the theorem, the hodograph $\chi(\lambda)$ for $|\lambda| = 1$ does not intersect the real axis on the interval $(-\infty, -\mu_0^{-1}]$, which, in view of the properties of the matrix P, is necessary and sufficient for the absolute stability of linear systems of the form (1) for $\varphi(\sigma) = \mu\sigma,\ 0 \le \mu \le \mu_0$. Thus, the spectra of all matrices P_μ (see (8)) lie inside the unit circle. By Lemma 3, the last inequality implies $H_1(\mu) > 0$. Therefore there exists $\varepsilon > 0$ such that $x^* H_1(\mu) x \ge \varepsilon |x|^2$ for $0 \le \mu \le \mu_0$. Then from (6), (8), for any $\varphi(\sigma)$ satisfying conditions (2), we easily obtain

$$V(x) = x^* H_1(\varphi/\sigma) x + 2\vartheta \left[\int_0^\sigma \varphi\, d\sigma - \frac{1}{2}\varphi\sigma\right] \ge \varepsilon |x|^2 - 2|\vartheta| \cdot \left|\int_0^\sigma \varphi\, d\sigma - \frac{1}{2}\varphi\sigma\right|$$

$$V(x) \ge \frac{\varepsilon}{2}|x|^2 - \text{const}, \qquad V(x) \to \infty \text{ as } |x| \to \infty.$$

Thus, function (6) satisfies all the conditions of Lemma 1, from which the assertion of the theorem follows.

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Note: Figure translations are in progress. See original paper for figures.