ON THE QUESTION OF COMPLETE CONTINUITY OF INTEGRAL OPERATORS

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Abstract

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ON THE QUESTION OF COMPLETE CONTINUITY OF INTEGRAL OPERATORS

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L. V. Kantorovich and B. Z. Vulikh gave in [1] (see also [3], Ch. VII, § 6) an abstract characterization of a broad class of integral operators, in particular Hilbert-Schmidt operators. In this note we characterize them again, but from a different point of view, namely as completely continuous in a certain strengthened sense. (For the necessary information on the theory of semi-ordered spaces, see [2, 3].)

We shall consider spaces consisting of real functions summable on $[0, 1]$, with the usual identification and ordering. Each of the spaces occurring below contains the function identically equal to one and is embedded normally in L (i.e., from $y \in X$, $|x| \le |y|$ it follows that $x \in X$). We shall consider the following classes of such spaces:

a) **Class P^-.** The spaces belonging to it are KB-linear spaces in which from $x_n \downarrow 0$ it always follows that $\|x_n\| \downarrow 0$.

b) **Class P.** In it we include those $X \in P^-$ which are KB-spaces.

c) **Class Q^-.** This is the class of spaces dual to spaces of the class P^-. $Y \in Q^-$ if and only if there exists an $X \in P^-$ such that the relations

$$m = \sup_{\|x\| \le 1} \int_0^1 x(t)y(t)\,dt < +\infty \quad \text{and} \quad y \in Y$$

are equivalent, and moreover $\|y\| = m$. Under these conditions Y will also be a KB-linear space, and the formula

$$f(x) = \int_0^1 x(t)y(t)\,dt, \quad y \in Y,$$

gives the general form of a linear functional in X. Henceforth the space dual to X will be denoted by $\bar{X}$.

d) **Class Q,** consisting of spaces dual to spaces of the class P.

It is known that the inclusion $X \in P \cap Q$ is equivalent to the reflexivity of X as a normed space, while the inclusion $X \in P \cup Q^-$ entails reflexivity in the sense of Nakano (i.e., $X = \bar{\bar{X}}$). An example of a space from Q^- may be any Orlicz space L^*_M.* If the convex function M satisfies the well-known Δ_2-condition, then $L^*_M \in P$. All spaces of type E_M are contained in P^-. If the function N complementary to M satisfies the Δ_2-condition, then $L^*_M \in Q$. Finally, we note that, as can be shown, the class $P \cap Q$ contains in large number spaces which do not coincide in their composition with any Orlicz space.

* Concerning Orlicz spaces, see [4].

In what follows we denote by S_X the unit sphere of the space X. We shall write $X_1 \subset X_2$ if all elements of the set S_{X_1} have uniformly absolutely continuous norms in X_2. The latter means that

$$\sup_{x \in S_{X_1}} \|x \cdot \chi_e\|_{X_2} \to 0 \quad \text{as} \quad me \to 0.$$

(Here χ_e is the characteristic function of the set e, and the dot denotes ordinary multiplication.) In other words, X_2 is essentially wider than X_1.

A set E lying in an arbitrary K-space Z will be called **strongly compact** if there exists $z > 0$ such that E is compact with respect to convergence with regulator z. More precisely: for any sequence $\{y_n\} \subset E$ one can indicate an element $y \in Z$ and a subsequence $\{y_{n_k}\}$, $n_1 < n_2 < \cdots$, such that for all $k = 1, 2, \ldots$ one has $|y - y_{n_k}| \le \lambda_k z$, where λ_k are real numbers and $\lambda_k \downarrow 0$. We emphasize that the regulator z is the same for all subsequences $\{y_n\} \subset E$.

We shall call an operator U, acting from a normed space X into a K-space Y, **strongly completely continuous***, if the image of the unit sphere $U(S_X)$ is strongly compact in Y. For spaces of the class $P - UQ^-$ it is clear that every strongly completely continuous operator will also be completely continuous in the usual sense, but not conversely. It turns out that the class of strongly completely continuous operators is closely connected with the class of so-called $(b-o)$-linear operators, an integral representation of which was obtained in the work (1). An operator U is called $(b-o)$-linear if $\|x_n\| \to 0$ implies $Ux_n \xrightarrow{(o)} 0$.

Let K be a measurable real function given in the square $0 \le s, t \le 1$. Consider the integral operator acting according to the formula

$$y(s) = \int_0^1 K(s,t) x(t)\, dt. \qquad (*)$$

In what follows we shall always denote it by U, independently of the spaces in which it is considered. It is easy to show that U is $(b-o)$-linear from $X \in P \cup Q$ into $Y \in P \cup Q^-$ if and only if the following conditions are satisfied:

1) For almost all $s \in [0,1]$, $K(s, \bullet) \in \bar{X}$ as a function of the second argument.

2) The function defined by the equality $y_0(s) = \|K(s, \bullet)\|_{\bar{X}}$ belongs to Y. These same conditions are sufficient for $(b-o)$-linearity in the case when $X, Y \in P - UQ^-$.

We can now formulate the main theorems of the present note.

Theorem 1. Let $Y \in P^-$, $X = Z$, $Z \in P^-$. In order that the integral operator defined by formula $(*)$ be strongly completely continuous from X into Y, it is sufficient that the following conditions be satisfied:

1′) For almost all $s \in [0, 1]$, $K(s, \bullet) \in Z$ as a function of the second argument.

2′) The function defined by the equality $y_0(s) = \|K(s, \bullet)\|_Z$ belongs to Y.

In the case when $Z \in P$, these conditions are also necessary for the strong complete continuity of the operator defined by equality $(*)$.

Theorem 2. If $X \in P \cap Q$, $Y \in P$, the classes of strongly completely continuous and $(b-o)$-linear operators coincide. The general form of such operators is given by formula $(*)$, where the function K, measurable in the square $0 \leq s, t \leq 1$, satisfies conditions 1) and 2).

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* The term is borrowed by us from (5), where the case is considered in which $U(S_X)$ is compact with respect to ordinary uniform convergence (i.e. $z(s) \equiv 1$).

We note that in the proof of this theorem the integral representation is established in the same way as was done in ([1]) for (b-o)-linear operators from L^p into L^q.

A strongly completely continuous operator is an operator which, for some $u_0 \in Y$, is completely continuous from X into the Banach space of bounded elements Y_{u_0}, the norm in which is introduced by the condition

$$\|z\| = \inf\{\lambda : \ |z| \leq \lambda u_0\}.$$

All the more, it will be completely continuous from X into Y.

The ordinary complete continuity of (b-o)-linear operators was noted earlier as well. Such a theorem was first proved in a paper of T. Ogasawara ([6]) (see also ([7])).

From Theorem 2, in particular, it follows that for $X = Y = L^2$ the strongly completely continuous operators are nothing other than the Hilbert-Schmidt integral operators characterized by the condition

$$\iint_{0 \leq s,t \leq 1} [K(s,t)]^2 \, ds \, dt < +\infty.$$

Thus such operators have again received an abstract description, different from that in ([1]), where they are characterized as (b-o)-linear.

Let us give several more criteria for complete continuity of integral operators with positive kernel, easily established with the aid of Theorem 1.

Theorem 3. *Let $X \in Q$, $Y \in P$. Each of the following five conditions is necessary and sufficient in order that the linear integral operator U with kernel $K \geq 0$ be completely continuous from X into Y.*

I. *U acts from some $X_1 \supset X$ into Y, $X_1 \in Q$.*

II. *U acts from X into some $Y_1 \subset Y$, $Y_1 \in P$.*

III. *U acts from some $X_1 \supset X$ into some $Y_1 \subset Y$, $X_1 \in Q$, $Y_1 \in P$.*

IV. *The integrals*

$$\iint_{e\times[0,1]} K(s,t)x(t)y(s)\,dt\,ds, \quad x \in S_X,\ y \in S_Y,$$

are uniformly absolutely continuous as functions of $e \subset [0,1]$.

V. *The integrals*

$$\iint_{[0,1]\times e} K(s,t)x(t)y(s)\,dt\,ds, \quad x \in S_X,\ y \in S_Y,$$

are uniformly absolutely continuous (in the same sense).

This theorem gives exact form to the well-known principle according to which complete continuity is "continuity with a margin."

Conditions IV and V may be replaced by the externally weaker, but in fact equivalent to each of them, condition VI:

VI.

$$\iint_{e_1\times e_2} K(s,t)x(s)y(t)\,dt\,ds \to 0 \quad \text{as } \mathrm{me}_1, \mathrm{me}_2 \to 0$$

uniformly with respect to $x \in S_X$, $y \in S_Y$.

With some modifications, the stated conditions of complete continuity remain valid also for the case $X \in O^-$, $Y \in P^-$. For Orlicz spaces, a criterion equivalent to condition VI was indicated by T. Ando ([8]).

Let us illustrate the application of Theorem 1, restricting ourselves here to one example, the situation of which is borrowed from the monograph ([4]). Let Φ be a convex function satisfying the usual requirements for the theory of Orlicz spaces, and also the known condition Δ'; let Ψ be complementary to it. Suppose that

$$\iint_{0\leq s,t\leq 1} \Psi(|K(s,t)|)\,dt\,ds < +\infty.$$

It follows from this, by means of simple estimates, that for almost all $s \in [0,1]$ one has $K(s,\cdot) \in L^*_\Psi$, and that the function defined by the equality $y_0(s) =$

$\|K(s,\cdot)\|_{L^*_\Psi}$ belongs to L^*_Ψ. Therefore the operator with kernel K will be (b-0)-linear from any $X \subset L^*_\Phi$ into any $Y \supset L^*_\Psi$. If, moreover, $X \in Q$, $Y \in P$, then it will also be completely continuous in the strong sense. This will be the case, for example, if $X = L^*_{M_1}$, $Y = L^*_{M_2}$, with $N_1 \prec \Psi$, $M_2 \prec \Psi$, and M_2 and N_1 satisfy the Δ_2-condition (M_i is complementary to N_i).

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