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Abstract

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MATHEMATICS

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THE CAUCHY PROBLEM FOR A LINEAR DIFFERENTIAL EQUATION OF INFINITE ORDER

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In the present article we shall consider the Cauchy problem for linear differential equations of infinite order with constant coefficients of the form

$$\sum_{n=0}^{\infty} a_n y^{(n)}(z) = f(z) \qquad (a_0 \neq 0) \tag{1}$$

and of the form

$$\sum_{n=0}^{\infty} a_n y^{(n)}(z) = 0 \qquad (a_0 \neq 0). \tag{2}$$

The right-hand side of equation (1)—the function $f(z)$—will be assumed analytic in some disk. We shall assume that the characteristic function of equations (1) and (2),

$$\varphi(t) = \sum_{n=0}^{\infty} a_n t^n,$$

is entire. Let us consider the formulation of the Cauchy problem for differential equations of infinite order.

Let N' be some proper subset of the set of all natural numbers with zero adjoined to it, i.e. of the set $\{0, 1, 2, 3, \dots, n, \dots\}$. It is required to find a solution if, at zero, the derivatives of all orders from the set N' are prescribed. For the existence of a solution the choice of the set N' must be made depending on the characteristic function $\varphi(t)$. In the article a special case of the Cauchy problem will be considered.

Let $\varphi(t)$ be an entire function of order of growth ρ and type σ, have infinitely many zeros, and satisfy conditions which we shall call **conditions A**:

1. $\varphi(t)$ has simple zeros $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \ldots$, with

$$|\lambda_1| < |\lambda_2| < |\lambda_3| < \cdots < |\lambda_n| < \cdots.$$

2. $\varphi(t)$ can be represented as a canonical product.

As the set N' we shall consider an arithmetic progression with difference ν,

$$N' = \{0, \nu, 2\nu, \ldots, n\nu, \ldots\}.$$

The number ν must satisfy conditions which we shall call **conditions B**:

1. ν must be such that

$$\sum_{j=1}^{\infty} \frac{1}{|\lambda_j|^{\nu}} < \infty.$$

2. We form the function

$$\psi(t^{\nu}) = \prod_{k=0}^{\nu-1} \varphi\big(te^{ik2\pi/\nu}\big) = \prod_{j=1}^{\infty} \left(1 - \frac{t^{\nu}}{\lambda_j^{\nu}}\right).$$

The function

$$\varphi_{\nu}(t) = \frac{\psi(t^{\nu})}{\varphi(t)} = \prod_{k=1}^{\nu-1} \varphi\big(te^{ik2\pi/\nu}\big)$$

is entire of order of growth ρ and type σ_{ν}, with $\sigma_{\nu} \leq (\nu - 1)\sigma$. The number ν must be such that $\sigma_{\nu} > 2\sigma$.

It can be proved that under conditions A and B there exists a certain perfect space of sequences ([1]) such that the Cauchy problem has a unique solution among the functions whose expansion coefficients in a power series belong to this space.

In Theorems 1-4 it is assumed that $\rho = 1,\ \sigma > 0$.

Theorem 1. *If the characteristic function $\varphi(t)$ and the number ν satisfy conditions A and B, respectively, then, for a right-hand side $f(z)$ analytic in the disk $|z| \leqslant \sigma_{\nu}$ and for the initial conditions $y^{(n\nu)}(0) = 0,\ n = 0, 1, 2, 3, \ldots$, there exists a number $\sigma_0,\ \sigma_{\nu} - \sigma \leqslant \sigma_0 \leqslant \sigma_{\nu} + \sigma$, such that the unique solution of equation (1) is analytic in the disk $|z| < \sigma_0$.*

The following theorem concerns a nonzero solution of the homogeneous equation (2).

Theorem 2. *If $\varphi(t)$ and ν satisfy conditions A and B, respectively, then, under the initial conditions*

$$y^{(n\nu)}(0) = 0, \qquad n = 0, 1, 2, 3, \dots,$$

there exists a number σ_0, $\sigma_\nu - \sigma \leqslant \sigma_0 \leqslant \sigma_\nu + \sigma$, such that a nonzero solution of the homogeneous differential equation (2) is analytic in a disk of radius not exceeding σ_0.

For nonzero initial conditions, their growth as a function of the number must satisfy certain restrictions.

Theorem 3. *If $\varphi(t)$ and ν satisfy conditions A and B, respectively, then, for a right-hand side $f(z)$ analytic in the disk $|z| \leqslant \sigma_\nu$, and for initial conditions $y^{(n\nu)}(0)$ such that the function*

$$y_\nu(z) = \sum_{n=0}^{\infty} \frac{y^{(n\nu)}(0)}{(n\nu)!} z^{n\nu}$$

is analytic in a disk of radius greater than $\sigma_\nu + \sigma$, there exists a number σ_0, $\sigma_\nu - \sigma \leqslant \sigma \leqslant \sigma_\nu + \sigma$, such that the unique solution of equation (1) is analytic in the disk $|z| < \sigma_0$.

Theorem 4. *If $\varphi(t)$ and ν satisfy conditions A and B, respectively, then, for initial conditions $y^{(n\nu)}(0)$ such that the function*

$$y_\nu(z) = \sum_{n=0}^{\infty} \frac{y^{(n\nu)}(0)}{(n\nu)!} z^{n\nu}$$

is analytic in a disk of radius greater than $\sigma^\nu + \sigma$, there exists a number σ_0, $\sigma_\nu - \sigma \leqslant \sigma_0 \leqslant \sigma_\nu + \sigma$, such that a nonzero solution of the homogeneous differential equation (2) is analytic in a disk of radius not exceeding σ_0.

The Cauchy problem can also be considered for equations in generalized derivatives. Let a sequence $(a_0, a_1, a_2, \dots, a_k)$, $a_k \neq 0$, $k = 0, 1, 2, 3, \dots$, be given. If $y(z) = \sum_{k=0}^{\infty} c_k z^k$, then the generalized derivative of the n-th order is defined by the equality

$$D^n y(z) = \sum_{k=n}^{\infty} \frac{\alpha_{k-n}}{\alpha_k} c_k z^{k-n}.$$

Equations in generalized derivatives can be represented in the form

$$\sum_{n=0}^{\infty} a_n D^n y(z) = f(z); \tag{3}$$

$$\sum_{n=0}^{\infty} a_n D^n y(z) = 0. \tag{4}$$

Theorem 5. If $\varphi(t) = \sum_{n=0}^{\infty} a_n t^n$ and the number ν satisfy, respectively, conditions A and B, and

$$\overline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\bar{\sigma}\rho)^{1/\rho}, \qquad \underline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\underline{\sigma}\rho)^{1/\rho},$$

then, for a right-hand side $f(z)$ analytic in the disk

$$|z| \leq (\sigma_\nu/\sigma)^{1/\rho},$$

and for the initial conditions

$$D^{n\nu} y(0) = 0, \qquad n = 0, 1, 2, 3, \dots,$$

there exists a number σ_0, $\sigma_\nu - \sigma \leq \sigma_0 \leq \sigma_\nu + \sigma$, such that the unique solution of equation (3) is analytic in the disk

$$|z| < (\sigma_0/\sigma)^{1/\rho}.$$

Theorem 6. If $\varphi(t)$ and the number ν satisfy, respectively, conditions A and B, and

$$\overline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\bar{\sigma}\rho)^{1/\rho}, \qquad \underline{\lim}_{n\to\infty} n^{1/\rho} \sqrt[n]{|\alpha_n|} = (\underline{\sigma}\rho)^{1/\rho},$$

then, under the initial conditions

$$D^{n\nu} y(0) = 0, \qquad n = 0, 1, 2, 3, \dots,$$

there exists a number σ_0, $\sigma_\nu - \sigma \leq \sigma_0 \leq \sigma_\nu + \sigma$, such that a nonzero solution of equation (4) is analytic in a disk of radius not greater than

$$(\sigma_0/\sigma)^{1/\rho}.$$

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CITED LITERATURE

1. R. Cook, *Infinite Matrices and Sequence Spaces*, Moscow, 1960.

Note: Figure translations are in progress. See original paper for figures.