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1965

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Abstract

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MATHEMATICS

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SOME CRITERIA FOR THE STABILITY OF FUNCTIONALS

(Presented by Academician S. L. Sobolev, 12 X 1964)

Suppose that a linear functional on the set of algebraic polynomials $\{P_n(x)\}$ is given by the segment of numbers

$$F(1),\ F(x),\ \dots,\ F(x^n). \tag{*}$$

Let this segment be not absolutely monotone ([1]). We denote its true nodes by $(\sigma_k)_1^{s_0}$ and the weights by $(\delta_k)_1^{s_0}$. Here always $2 \le s_0 \le n+1$; among the (σ_k) there are no equal ones, and among the (δ_k) there are no zeros.

Put

$$R_{s_0}(x) = \prod_1^{s_0}(x - \sigma_k)$$

—the resolvent of the functional $(*)$, and

$$R_{s_0,m}(x) = \prod_{k \ne m}(x - \sigma_k).$$

The weights (δ_k) are determined from the compatible system of $n+1$ equations

$$\sum_{k=0}^{s_0} \delta_k \sigma_k^m = F(x^m) \qquad (m = 0, 1, \dots, n) \tag{1}$$

by the formulas

$$\delta_m = \frac{F(R_{s_0,m})}{\prod_{i \ne m}(\sigma_m - \sigma_i)}. \tag{2}$$

Choose on $[0,1]$ arbitrarily s mutually distinct points; call them **conditional nodes**, and

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$$\overline{R}_s(x) = \prod_1^s (x - \rho_i)$$

the **conditional resolvent**; let

$$\overline{R}_{s,m}(x) = \prod_{i \neq m} (x - \rho_i).$$

From the first s equations of the system ($s \leq n + 1$)

$$\sum_{i=0}^{s} \overline{\delta}_i \rho_i^m = F(x^m) \qquad (m = 0, 1, \dots, n) \tag{3}$$

we have the conditional weights

$$\overline{\delta}_m = F(\overline{R}_{s,m}) \Big/ \prod_{i \neq m} (\rho_m - \rho_i). \tag{4}$$

In contrast to system (1), system (3) is in general incompatible when $s < n + 1$.

In the particular case where $s = s_0$ and all $(\rho_i) = (\sigma_k)$, respectively, then $(\overline{\delta}_i) = (\delta_k)$ and the entire system (3) is compatible. Conversely, if for $s = s_0$ all $\overline{\delta}_i = \delta_i$ and system (3) is compatible, then the system of nodes (ρ_i) coincides with (σ_k).

We note some consequences of the formulas written above.

Corollary 1. If $s_0 < s$ ($\leq n + 1$), then one can always choose the conditional nodes $(\rho_i)_1^s$ so that exactly $l = s - s_0$ of the nodes have zero loads in formulas (4). For this it is sufficient that $(\sigma_k)_1^{s_0} \subset (\rho_i)_1^s$.

Corollary 2. If, for no $(\rho_i)_1^s$, it is possible to turn in system (4), for $m = s$, at least one load δ into zero, then certainly $s_0 \geq s$.

Corollary 3. If, although it is possible under the conditions of the corollary to turn some δ into zero, this is possible only for such ρ that certainly cannot be a system of true nodes of the given functional, then even in that case $s_0 \geq s$.

Theorem. A sufficient condition for the number s_0 of true nodes (*) to satisfy the condition $s_0 \geq s$, where s is chosen in advance ($s \leq n + 1$), consists in the inconsistency, for all $(\rho_i)_1^s$, of the system of conditions, numbering $n + 2 - s$,

$$F(\overline{R}_s) = 0, \quad F(x\overline{R}_s) = 0, \dots, \quad F(x^{n-s}\overline{R}_s) = 0. \tag{a}$$

$$F(\overline{R}_{s,m}) = 0, \qquad (\ )$$

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where m is equal to one of the numbers $1, 2, \dots, s$.

Indeed, if $s_0 < s$ and (ρ_i) are chosen so that they contain (σ_k), then $\overline{R}_s(x) = R_{s_0}(x)r(x)$, and the conditions (a) are necessarily fulfilled. As for condition ( ), it is the requirement, following from (4), that at least one of the loads $\delta = 0$. The impossibility of satisfying conditions (a) and ( ) simultaneously indicates that $s_0 \geq s$.

Remark. According to Corollary 3, the theorem is also valid under smaller restrictions: if conditions (a) and ( ) are satisfied jointly only for such sets $(\rho_i)_1^s$ that certainly cannot be a set of true nodes of degree n (1), then in this case too $s_0 \geq s$.

We shall call a functional F, depending on some parameter ξ, **stable in the interval** $\alpha < \xi < \beta$, if in this interval the number of its true nodes $s_0(\xi) \geq n$.

We give examples of functionals whose stability is investigated with the aid of the criterion proved above.

Example 1. $F = 0_0, 0_1, \dots, 0_{n-1}, \xi$ $(\xi \neq 0)$. Put $s = n + 1$. We have one condition ( ): $F(R_{n+1,m}) = 0$, i.e. $\xi = 0$. Thus, F has $n + 1$ true nodes for all $\xi \neq 0$.

Example 2. $F = 0_0, 0_1, \dots, 0_{n-2}, 1_{n-1}, \xi$; let $s = n + 1$. Owing to the arbitrariness of (ρ_i) on $[0, 1]$, for $\xi \leq 0$ and for $\xi \geq n$ the condition is clearly not fulfilled; consequently, $s_0 = n + 1$. Put $s = n$; then we have two conditions: $F(\overline{R}_n) = 0$, $F(\overline{R}_{n,m}) = 0$. They give $\xi = s_1$ and $1 = 0$. Thus, the given functional has $s_0 \geq n$ for all real ξ.

Example 3. $F_\xi(P_n) = P_n^{(k)}(\xi)$ $(1 \leq k \leq n)$ (2). Put $s = n + 1$. Condition ( ) gives

$$R^{(k)}_{n+1,m}(\xi) = 0. \tag{5}$$

The impossibility of this requirement for $\xi \geq 1$ or $\xi \leq 0$ is obvious, since all roots of equation (5) lie inside $[0, 1]$.

Put $s = n$. We have two conditions: $R_n^{(k)}(\xi) = 0$, $\overline{R}_{n,m}(\xi) = 0$. Denote $\overline{R}_{n,m}(\xi) = Q_{n-1}(\xi)$; then $\overline{R}_n(\xi) = Q_{n-1}(\xi)(\xi - \rho_m)$, and the system of conditions takes the form $Q^{(k)}_{n-1}(\xi) = 0$; $Q^{(k-1)}_{n-1}(\xi) = 0$, which is inconsistent for simple roots. Thus, $s_0 \geq n$ for all ξ.

Example 4. 1) $F_{\rho,\xi}(P_n) = \operatorname{Re} P_n(z)$; 2) $F_{\rho,\xi}(P_n) = \operatorname{Im} P_n(z)$, i.e. 1) $F_{\rho,\xi}(x^k) = \rho^k \cos k\xi$; 2) $F_{\rho,\xi}(x^k) = \rho^k \sin k\xi$ $(k = 0, 1, \dots, n)$. Let $s = n$; then we have: 1) $\operatorname{Re} \overline{R}_n(z) = 0$, $\operatorname{Re} \overline{R}_{n,m}(z) = 0$, or, putting $\overline{R}_{n,m}(z) = Q_{n-1}(z)$, $\operatorname{Re} Q_{n-1}(z) = 0$, $\operatorname{Re}[Q_{n-1}(z)(z - \rho)] = 0$. These requirements are compatible only for real z.

The same result is obtained in case 2).

Thus, off the axis Ox one has $s_0 \geq n$ (3).

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Example 5. $F_x = 1, \ \dfrac{x}{2}, \ \dfrac{x^2}{3}, \dots, \dfrac{x^n}{n+1}$ is an integral functional on $\{P_n(x)\}$, since

$$F_x(P_n) = \frac{1}{x}\int_0^x P_n(\xi)\,d\xi.$$

It is obvious that for $x < 0$ the number of true nodes is $s_0 = n + 1$, while for $0 < x \leq 1$ the functional is absolutely monotone (amorphous) (1).

Before applying the criterion to the case $x > 1$, we shall have to introduce some auxiliary estimates.

Lemma. Let $f(x)$ be continuous on $[0, 1]$ and $0 < \alpha < 1$. Then, after replacing α by one of the numbers 0 or 1 in

$$I(\alpha) = \int_0^1 (x - \alpha) f(x)\,dx$$

the modulus of the integral will not decrease, i.e. $I(\alpha)$ is majorized either by

$$\left|\int_0^1 x f(x)\,dx\right|,$$

or by

$$\left|\int_0^1 (1 - x) f(x)\,dx\right|.$$

Indeed,

$$I(\alpha) = \int_0^1 x f(x)\,dx - \alpha\int_0^1 f(x)\,dx.$$

1) Let $I(\alpha) > 0$. Then, if $\int_0^1 f(x)\,dx > 0$, then

$$0 < I(\alpha) < \int_0^1 x f(x)\,dx,$$

and if $\int_0^1 f(x)\,dx < 0$, then

$$0 < I(\alpha) < \int_0^1 (x - 1) f(x)\,dx.$$

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2) Let $I(\alpha) < 0$; if $\int_0^1 f(x)\,dx > 0$, then

$$\int_0^1 (x-1)Q(x)\,dx < I(\alpha) < 0;$$

whereas if $\int_0^1 f(x)\,dx < 0$, then

$$\int_0^1 xQ(x)\,dx < I(\alpha) < 0.$$

Thus, the lemma is proved.

Corollary. For $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$, in

$$\int_0^1 \prod_1^n (x-\alpha_i)\,dx$$

each of the numbers α_i may be replaced by one of the numbers 0 or 1 in such a way that

$$\left|\int_0^1 \prod_1^n (x-\alpha_i)\,dx\right| \le \int_0^1 x^k(1-x)^{n-k}\,dk.$$

Equality in this estimate occurs only if $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ and $\alpha_{k+1} = \cdots = \alpha_n = 1$.

Remark 1. The beta integral

$$\int_0^1 x^k(1-x)^{n-k}\,dx = B_n(k) \qquad (0 \le k \le n)$$

has a maximum at $k = 0$ and $k = n$ and a minimum at $k = E(n/2)$. Thus,

$$B_n(k) \le \frac{1}{n+1}.$$

Remark 2. Denote

$$\int_\alpha^\beta \xi^k(\xi-1)^{n-k}\,d\xi = I_{k,n-k}(\alpha,\beta);$$

we have the directly verified recurrence formula

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$$I_{k+1,n-k-1}(\alpha - \beta) - I_{k,n-k}(\alpha, \beta) = I_{k,n-1-k}(\alpha, \beta). \qquad (6)$$

We return to the integral functional, putting in it $x > 1$. The general form of conditions (a) and (b) here reduces to the conditions

$$\int_0^x \xi^k \overline{R}_s(\xi)\, d\xi = 0 \quad (k = 0, 1, \dots, s-k); \qquad \text{(a')}$$

$$\int_0^x \overline{R}_{s,m}(\xi)\, d\xi = 0. \qquad \text{(b')}$$

Let $s = n+1$. We have one condition

$$\int_0^x \overline{R}_{n+1,m}(\xi)\, d\xi = 0$$

or

$$\int_0^1 \prod_1^n (\xi - \rho_i)\, d\xi + \int_1^x \prod_1^n (\xi - \rho_i)\, d\xi = 0,$$

briefly,

$$I_1 + I_2(x) = 0;$$

but

$$|I_1| \leqslant \int_0^1 x(1-x)^{n-1}\, dx = \frac{1}{n(n+1)}$$

(see Corollary 3);

$$I_2(x) > \int_1^x (\xi - 1)^n\, d\xi = \frac{(x-1)^{n+1}}{n+1}.$$

Thus, the condition is certainly impossible if

$$(x-1)^{n+1} > \frac{1}{n}.$$

For all $n = 2, 3, \dots$ and $x \geqslant 2$ we have $s_0 = n+1$.

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Let $s = n$. We have two conditions which, if we put $R_{n,m}(\xi) = Q_{n-1}(\xi)$, may be written in the form

$$\int_0^x Q_{n-1}(\xi)\,d\xi = 0, \qquad \int_0^x \xi Q_{n-1}(\xi)\,d\xi = 0.$$

We find conditions for the certain impossibility of the second, as the stronger one:

$$\int_0^1 \xi \prod_1^{n-1}(\xi - \rho_i)\,d\xi + \int_1^x \xi \prod_1^{n-1}(\xi - \rho_i)\,d\xi = I_1 + I_2(x),$$

$$|I_1| \leqslant \int_0^1 \xi(1-\xi)^{n-1}\,d\xi = \frac{1}{n(n+1)},$$

$$I_2(x) > \int_1^x \xi(\xi-1)^{n-1}\,d\xi = (x-1)^n \frac{nx+1}{n(n+1)}.$$

Thus, for violation of the condition it is sufficient that the inequality

$$(x-1)^n > \frac{1}{nx+1} \qquad (1 < x < 2).$$

be satisfied.

Suppose

$$x = 2 - \frac{p}{n} \quad (p < n \text{ and const}).$$

We have

$$\left(1 - \frac{p}{n}\right)^n > \frac{1}{2n+1-p};$$

then for $p = 1$ and $n > 2$ the inequality is satisfied and $s_0 \geqslant n$ for

$$x \geqslant 2 - \frac{1}{n}.$$

For $p = 2$ and $n \geqslant 6$, also $s_0 \geqslant n$ for

$$x \geqslant 2 - \frac{2}{n},$$

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and so on.

In conclusion, we note that the integral functional in any interval $(2-\varepsilon, 2)$, where $\varepsilon > 0$ is an arbitrarily small constant, is not stable for sufficiently large n.

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Received
12 IX 1964

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Note: Figure translations are in progress. See original paper for figures.

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