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**Abstract**

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## Astronomy

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# On One Mechanism of Instability for Pulsating Stars

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The paper considers the question of the possibility of amplification of fundamental modes of nonradial natural oscillations of a compressible gravitating sphere undergoing oscillations of the type of radial pulsations. For simplicity, the study is carried out for the case of a homogeneous distribution of the density of the medium, which should not be an essential restriction for the indicated mechanism of instability. The initial system of equations is written in the form

$$\rho\, d\mathbf{v}/dt = -\nabla p - \rho\nabla\Phi, \qquad \nabla^2\Phi = 4\pi G\rho, \qquad d\rho/dt = -\rho\,\operatorname{div}\mathbf{v}, \qquad (1)$$

$$\frac{d}{dt}\,(p\rho^{-\gamma}) = 0, \qquad d/dt = \partial/\partial t + (\mathbf{v}\cdot\nabla), \qquad \gamma = \text{const.}$$

Let us first consider the radial pulsations of the sphere. Introducing Lagrangian coordinates $a = (r)_{t'=0}$, $t' = t$, we obtain (below, everywhere, instead of $t'$ we write $t$)

$$\rho r^2 \partial r/\partial a = \rho_0 a^2, \qquad p/p_0 = (\rho/\rho_0)^\gamma, \qquad v_r = \partial r/\partial t, \qquad v_\vartheta = v_\varphi = 0, \quad (2)$$

$$\frac{\partial^2 r}{\partial t^2} = -\left(\frac{\partial r}{\partial a}\right)^{-1}\left(\frac{1}{\rho}\frac{\partial p}{\partial a} + \frac{\partial\Phi}{\partial a}\right) = -\frac{r^2}{\rho_0 a^2}\frac{\partial p}{\partial a} - \frac{4\pi G}{r^2}\int_0^a \rho_0(a)a^2\,da, \quad (3)$$

$$\rho = \rho(a,t), \qquad \rho_0 = \rho(a,0), \quad \text{etc.}$$

Here $r, \vartheta, \varphi$ are spherical coordinates.

Now substitute into (1), instead of $\rho, \Phi$, etc., $\rho(a,t) + \rho^*(a,\vartheta,\varphi,t)$, $\Phi(a,t) + \Phi^*(a,\vartheta,\varphi,t)$, etc., where the asterisk denotes perturbations, and linearize this system. In the coordinates $a, \vartheta, \varphi, t' = t$ we shall have

$$\rho\left(\frac{\partial r}{\partial a}\frac{\partial v_r^*}{\partial t}+\frac{\partial v_r}{\partial a}v_r^*+\frac{\partial \Phi^*}{\partial a}\right)-\frac{\rho^*}{\rho}\frac{\partial p}{\partial a}+\frac{\partial p^*}{\partial a}=0; \tag{4}$$

$$\frac{\partial r}{\partial a}\left(\frac{\partial \rho^*}{\partial t}+\rho\,\mathrm{div}\,\mathbf{v}^*\right)+\frac{\rho^*}{r^2}\frac{\partial r^2 v_r}{\partial a}+\frac{\partial \rho}{\partial a}v_r^*=0; \tag{5}$$

$$\frac{\partial r}{\partial a}\left(\frac{\partial p^*}{\partial t}+\gamma p\,\mathrm{div}\,\mathbf{v}^*\right)+\frac{\gamma p^*}{r^2}\frac{\partial r^2 v_r}{\partial a}+\frac{\partial p}{\partial a}v_r^*=0 \tag{6}$$

and three more equations following from the first two equations of system (1).

Let the density in the steady state be $\rho_0=\rho_{00}=\mathrm{const}$ for $a\le R-\delta$, $\delta\ll R$, and in the thin layer $R-\delta\le a\le R$ let the function $\rho_0(a)$ decrease sharply from $\rho_{00}$ to zero. For this case the solution of (3) of the form $r=aw(t)$ is known ([1–3]). If it is assumed that at the moment $t=0$ the sphere passes through the equilibrium position and $v_r(a,0)>0$, then the distributions of the quantities in the region $a\le R-\delta$, in accordance with (2)–(3), will be

$$r=aw(t),\qquad v_r=a\dot w,\qquad \dot w=\frac{dw}{dt},\qquad w(0)=1,\qquad u=(\dot w)_{t=0}>0, \tag{7}$$

$$\rho=\rho_{00}w^{-3},\qquad p=p_0w^{-3\gamma},\qquad p_0=\frac{1}{2}\rho_{00}\Omega^2(R^2-a^2),$$

$$\ddot w=\Omega^2\left(w^{-3\gamma+2}-w^{-2}\right),\qquad \Omega^2=\frac{4\pi}{3}G\rho_{00}. \tag{8}$$

Integrating (8), we find

$$t=\int_1^w\frac{dw}{\sqrt{u^2+2\Omega^2w^{-1}-\frac{2\Omega^2}{3(\gamma-1)}\left(w^{-3\gamma+3}+3\gamma-4\right)}}. \tag{9}$$

In order that a periodic solution $w(t)$ exist, it is necessary to assume [3] $3\gamma>4$.

The system of equations of the stability problem (4)–(6), etc., in the region of uniform density is satisfied if

$$\mathbf{v}^*=wV(t)\,\mathrm{grad}(a^lY_l^m),\quad p^*=\rho_{00}w^{-2}Q(t)a^lY_l^m,\quad \rho^*=0, \tag{10}$$

$$\Phi^*=w\Psi(t)a^lY_l^m,\quad Y_l^m=P_l^{|m|}(\cos\vartheta)e^{im\varphi},\quad m=0,\pm1,\dots,\pm l,$$

and the functions of time are related by

$$\dot{V} + w^{-1}uV + Q + \Psi = 0, \quad \dot{Q} + (3\gamma - 2)w^{-1}uQ - l\Omega^2 w^{-3\gamma+1}V = 0. \tag{11}$$

This solution is not complete; nevertheless, it makes it possible to investigate those oscillations which do not vanish in the limiting transition to the case of an incompressible medium.

Let us now consider the conditions following from integration of the original system in the region $R-\delta \le a \le R$, where $p(a,t)$ is small and $\rho(a,t)$ is a sharply varying function of $a$. One may assume that, to accuracy up to corrections $\delta/R$, the following quantities in the layer are functions of $t$ alone: $r$, $v_r$, $\partial r/\partial a$, and $\partial v_r/\partial a$. It follows from (3) that the ratio $(1/\rho)(\partial\rho/\partial a)$ is such a quantity as well. Multiplying equations (4)–(6) and the others by $da$ and integrating from $R-\delta$ to $R$, we obtain

$$p^*\big|_{R-\delta}^{R} - \frac{1}{\rho}\frac{\partial p}{\partial a}S = 0, \quad S = \int_{R-\delta}^{R} \rho^*(a,\vartheta;\varphi,t)\,da; \tag{12}$$

$$\left.\frac{\partial \Phi^*}{\partial a}\right|_{R-\delta}^{R} - 4\pi G\left(\frac{\partial r}{\partial a}\right)^2 S = 0, \quad \Phi^*\big|_{R-\delta}^{R} = 0; \tag{13}$$

$$-\frac{\partial r}{\partial a}\frac{\partial S}{\partial t} + \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial a}S + \rho v_r^*\big|_{R-\delta}^{R} = 0. \tag{14}$$

From the original system and (14) it further follows that

$$\int_{R-\delta}^{R} \mathbf{v}^*\,da \sim \mathbf{v}^*\delta, \quad \int_{R-\delta}^{R} p^*\,da \sim p^*\delta.$$

Integration of (14) gives $S = \rho_{00}\Omega^{-2}w^{3\gamma-5}QR^{l-1}Y_l^m + C_0 w^{-3}$, where, in view of (12), the constant $C_0 = 0$. Using the expression for $\Phi^*$ in the region $a \ge R$, $\Phi^* = \psi R^{2l+1}r^{-l-1}Y_l^m$, with the aid of (13) we find that $(2l+1)\Psi = -3Qw^{3\gamma-4}$. If we also put $Q = w^{-3\gamma+1}y$, then from (11) we obtain an equation for $y(t)$,

$$d^2y(t)/dt^2 + \omega^2[1 - f(t)]y(t) = 0; \tag{15}$$

$$\omega = \Omega\sqrt{2l(l-1)/(2l+1)}, \quad f = \frac{1}{2l}\left[w^{-3} - (2l+1)w^{-3\gamma+1} + 2l\right]. \tag{16}$$

The character of the time dependence of the perturbations is determined by the Hill-type equation (15). This means that the configuration under consideration, for $l \ge 2$, may have unstable normal oscillations with amplitude increasing without bound as $t \to \infty$. If pulsations are absent, then the solutions of (15) are harmonic oscillations with angular frequency $\omega$. For the case of an incompressible sphere such solutions were studied in work [4].

We shall carry out the investigation of the instability regions of equation (15) for the example of pulsations of small amplitude. Using (8), (9), and (16), we ob-

of the expansion

$$w(t) = 1 + \varepsilon \sin \sigma t + \frac{\varepsilon^2 (3\gamma + 1)}{12} (3 - 4\cos \sigma t + \cos 2\sigma t) + O(\varepsilon^3); \qquad (17)$$

$$\sigma^2 = (3\gamma - 4)\Omega^2 \left[1 - \frac{\varepsilon^2}{24}(18\gamma^2 + 21\gamma - 13) + O(\varepsilon^3)\right], \qquad \varepsilon = \frac{u}{\Omega\sqrt{3\gamma - 4}}; \quad (18)$$

$$f(t) = \varepsilon h \sin \sigma t + \varepsilon^2 (\gamma + 1/3) h (1 - \cos \sigma t) -$$

$$-\varepsilon^2 q (1 - \cos 2\sigma t) + O(\varepsilon^3); \qquad (19)$$

$$h = 3\gamma - 1 + \frac{1}{2l}(3\gamma - 4), \qquad q = \frac{1}{12}\left[(3\gamma - 1)(12\gamma + 1) + \frac{1}{l}(3\gamma - 4)(6\gamma + 5)\right].$$

Here $u = (\dot{w})_{t=0} > 0$, $0 < \varepsilon \ll 1$. Substituting the expression for $f(t)$ into equation (15), we arrive at an equation for which the theory of asymptotic expansions has been developed [5]. Solutions growing without bound as $t$ increases are possible only near certain resonant values of $\omega$, when

$$2\omega \approx k\sigma \quad \text{or} \quad 8l(l-1)/(2l+1) \approx k^2(3\gamma - 4), \qquad k = 1, 2, \ldots \qquad (20)$$

Assuming that $|(2\omega/k\sigma) - 1| \ll 1$, we shall seek the function $y(t)$ in the form [5] $y(t) = x(t)\cos\left[\frac{1}{2}k\sigma t + \psi(t)\right] + \ldots$, $|\dot{x}| \ll \sigma|x|$, $|\dot{\psi}| \ll \sigma|\psi|$. For a solution that can become unstable, we find that the amplitude $x(t)$ is equal to $\exp(\varepsilon^k \lambda_k t)$, where

$$\lambda_1 = \sqrt{\frac{h^2\omega^4}{4\sigma^2} - \frac{1}{\varepsilon^2}\left(\omega - \frac{\sigma}{2}\right)^2},$$

$$\lambda_2 = \frac{\omega^2}{2\sigma}\sqrt{\frac{1}{4}\left(\frac{\omega^2 h^2}{2\sigma^2} - q\right)^2 - \left[\frac{1}{\varepsilon^2}\left(1 - \frac{\sigma^2}{\omega^2}\right) - \frac{\omega^2 h^2}{6\sigma^2} - \left(\gamma + \frac{1}{3}\right)h + q\right]^2}. \quad (21)$$

In the region of instability $\lambda_k > 0$.

The occurrence of the instabilities considered is explained by the possibility of transferring the energy of radial pulsations into other types of oscillations. One may expect that, for stars of variable density (pulsating with small amplitude), the instability condition (20) will be written in the form $2T_0 \approx kT_l,\ k = 1, 2, ...$, where $T_l$ is the period of the corresponding natural oscillation. For a model close to the standard one, at $k = 1$ nonradial oscillations of $g$-modes with small $l$ may become unstable.

The question of the character in which the developing instability manifests itself is very complicated. A decrease in the amplitude of the perturbation could occur, for example, as a result of the ejection of matter. It is possible that the abrupt changes of radial velocities observed in Cepheids and other pulsating stars ([3]) are a manifestation of the instability considered here. An estimate, by formulas (21), of the maximum relative increase of the perturbation amplitude over a pulsation period for $k = 1,\ \gamma = 5/3$ gives $\Delta = [y(2\pi/\sigma)/y(0)]_{\max} \approx \exp(\varepsilon\pi)$. If $\varepsilon = 0.1$, then $\Delta = 1.37$; for $\varepsilon = 0.22$ the amplitude of the perturbation over the period $2\pi/\sigma$ doubles. Obviously, the amplitude of nonradial oscillations will also depend on how much the perturbation is reduced as a result of the ejection of matter or some other mechanism for removing the instability.

For sufficiently large $\varepsilon$, the influence of the developing instability may be so substantial that the period and amplitude of the initial radial pulsations will change depending on the amplitude of the perturbation. This phenomenon could be the cause of period variation in some variable stars. Let us consider, for example, stars of the RV Tauri type. It is known that the variability of these stars could be explained by the combined influence of pulsations and instability ([6]). If the latter is the result of resonance $k = 1$, then one must assume that the fundamental period is equal to the pulsation period, while the formal period twice as large corresponds–

corresponds to the most rapidly developing nonradial oscillations. With this explanation, it becomes clear why the fundamental period connects stars of the RV Tauri type with Cepheids and with long-period variables by a common relation between period and spectral class ([6]).

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*Note added in proof.*

For the case of small pulsation amplitudes and a nonuniform density distribution $\rho_0(a)$, the solution of the system of equations of the stability problem (4), (5), (6), $\cdots$can be obtained by the method of perturbation theory. Investigation of the resonance of the first approximation ($k = 1$) leads to the following result. If nonradial oscillations exist whose doubled frequency is close to the frequency of the radial pulsations, there is a solution for the amplitude of the nonradial oscillations containing the factor $\exp(\varepsilon\lambda_1 t)$. Here $\varepsilon$ is a certain mean relative pulsation amplitude; $\lambda_1$ is determined by the first formula (21), in which the constant $h$ depends on the stellar model. In this case $h^2 \geq 0$; therefore the conclusions of the present work remain valid for a nonuniform distribution of the density of the medium.

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*Note: Figure translations are in progress. See original paper for figures.*