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# Mathematics

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**Abstract**

**Full Text**

**Mathematics**

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## On the Optimal Cubature Lattice for Completely Smooth Functions of Two Variables

Let the number of measurements  $n$  and a natural number  $m > n/2$  be given. Then the series

$$\sum \frac{1}{r^{2m}}, \quad (1)$$

where  $r$  denotes the distances from the zero point  $O$  of a certain  $n$ -dimensional point lattice  $\Gamma$  to all its other points, converges absolutely. Let its sum be equal to  $S_{m,\Gamma}$ . We shall call a lattice for which, among all  $n$ -dimensional lattices with the given volume  $v$  of the fundamental parallelepiped, the sum  $S_{m,\Gamma}$  is smallest, **optimal** for the given  $m$ . It is obvious that the property of a lattice of being optimal in this sense is invariant with respect to similarity transformations. S. L. Sobolev showed that for smooth functions of  $n$  variables for which, over the domain of integration, the sum of the squares of all partial derivatives of order  $m$  is bounded, the cubature lattice with the given volume  $v$  of the fundamental parallelepiped that gives the least value of the remainder term is the lattice  $\Gamma^*$ , reciprocal to the optimal one. It was important to construct a general theory of optimal lattices for arbitrary  $n$  and  $m > n/2$ . In the present note we give a proof of the theorem:

**Theorem.** *The two-dimensional optimal lattice for any  $m \geq 2$  is the lattice  $\Gamma_1^2$  constructed on the regular triangle.*

The proof of this theorem proposed below apparently already contains some ideas that may form the basis of a general theory.

### 1°. Local extremality of the regular triangle.

**Lemma 1** (A. F. Gametskii and N. N. Sandakova). *The sum  $\sum 1/r^{2m}$ , for any  $m \geq 2$ , for the vertices of a regular triangle with center at the point  $O$ , is minimal with respect to all triangles obtained from it by an equiaffine transformation sufficiently little different from the identity and leaving the point  $O$  fixed.*

The proof is carried out by the usual means of differential calculus.

Fig. 1

Figure 1: Fig. 1

Since all points of the lattice  $\Gamma_1^2$  can evidently be divided into pairwise nonintersecting triples of vertices of regular triangles with center at the point  $O$ , and the series (1) converges absolutely and therefore can be split into such triple sums, it follows at once from this lemma that the lattice  $\Gamma_1^2$  is locally minimal for any  $m$ . This last fact is not difficult to prove directly by differentiating the whole series. However, Lemma 1 does not yet imply the optimality of the lattice  $\Gamma_1^2$ .

**2°. On the constant  $\lambda_C$ .**

**Lemma 2** (N. N. Sandakova). *Let an  $n$ -dimensional lattice  $\Gamma$  be given, the minimum distance between whose points is  $a$ . Then, if for an equiaffine lattice  $\Gamma'$  the minimum  $a'$  is sufficiently small, i.e.  $a' = \lambda a$ , where  $\lambda$  is less than some  $\lambda_C < 1$ , then certainly  $S_{m,\Gamma'} > S_{m,\Gamma}$ .*

A trivial estimate of the number  $\lambda_C$  is obtained if one assumes that the contribution to  $S_{m,\Gamma'}$  from the points of a linear row constructed on the minimum of the lattice  $\Gamma'$  is already greater than  $S_{m,\Gamma}$ , i.e., that  $\zeta(2m)/a'^{2m} > S_{m,\Gamma}$ . But, as N. N. Sandakova showed, the number  $\lambda_C$  can be estimated much more accurately. Thus, for example, for the lattice  $\Gamma = \Gamma_1^2$  and  $m = 2$ , the trivial estimate just described gives  $\lambda_C > 0.726$ , while by proceeding somewhat more carefully one can show that  $\lambda_C > 0.8579$ .

Let us give a method by which this value can be obtained. For this purpose, first compute the sum  $S_{m,\Gamma_1^2}$  for the lattice  $\Gamma_1^2$  with side of the triangle equal to one. The usual method of analytic number theory gives the formula

$$S_m = S_{m,\Gamma_1^2} = 6\zeta(m) \frac{1}{1+2^{-m}} \left[ 1 - \sum_{k=1}^{\infty} \left( \frac{1}{(6k-1)^m} - \frac{1}{(6k+1)^m} \right) \right].$$

**Fig. 1**

Computing by this formula, we obtain  $S_2 = 7.71148\dots$  and  $6.38 > S_3 > S_4 > \dots > S_m > \dots > 6$ .

Now let us proceed to finding the number  $\lambda_C$  for the lattice  $\Gamma_1^2$  and arbitrary  $m \geq 2$ . The minimum of the lattice  $\Gamma_1^2$  is equal to one. Let the lattice  $\Gamma'$  have minimum  $\lambda$ . In order to cover all points of the lattice  $\Gamma'$ , it suffices to consider all its primitive vectors (i.e., vectors issuing from the point  $O$  and containing no lattice points inside them), write the summands  $1/r^{2m}$  for all their endpoints, and multiply the resulting sum by  $2\zeta(2m)$ . Let  $OA$  be the shortest vector, i.e., of length  $\lambda$ , of the lattice  $\Gamma'$ . We shall call the linear row constructed on it

the zero-th row; the nearest row parallel to it we shall call the first row, and so on (see Fig. 1). Let  $2\zeta(2m)S_I$  be a number not exceeding the contribution to  $S_{m,\Gamma'}$  from the points of the first row (which are all primitive); in that case the contribution from the primitive points of the  $k$ -th row, as is easy to see, is no less than  $2\zeta(2m)S_I\varphi(k)/k^{2m}$ , where  $\varphi(k)$  is Euler's function. Therefore the whole sum  $S_{m,\Gamma'}$  is no less than

$$\frac{1}{\lambda^{2m}}2\zeta(2m) + 2\zeta(2m)S_I \times \sum_{k=1}^{\infty} \frac{\varphi(k)}{k^{2m}},$$

or, since

$$\sum_{k=1}^{\infty} \frac{\varphi(k)}{k^l} = \zeta(l-1),$$

we have

$$S_{m,\Gamma'} \geq \frac{1}{\lambda^{2m}}2\zeta(2m) + 2\zeta(2m-1)S_I.$$

To estimate the sum  $S_I$  from below, one must consider the indicated edge  $AOB$  of the lattice  $\Gamma'$  and the corresponding "angle"  $BOC$ . If  $OA = \lambda$ , then the point  $B$  lies on the segment  $\beta = (0 \leq x \leq \lambda/2, \sqrt{3}/2\lambda)$ , and the point  $C$  on the segment  $\gamma = (-\lambda \leq x \leq -\lambda/2, \sqrt{3}/2\lambda)$ . Clearly, the whole contribution to  $S_I$  from the point  $B$  and the points of the first row lying to its right will be least if the point  $B$  is located at the right endpoint of the segment  $\beta$ , and the contribution from the point  $C$  and the points of the first row lying to its left will be least if the point  $C$  is located at the left endpoint of the segment  $\gamma$ . (As can be shown, for  $m = 2$  the estimate obtained in this way can be improved by choosing the point  $B$  at the left endpoint of the segment  $\beta$ , and taking as  $C$  the point with coordinates  $(-3\lambda/2, \sqrt{3}/2\lambda)$ , which already gives us the value  $\lambda_C = 0.8579$ .)

A more accurate method of estimation is the following. Divide the segment  $\beta$  into small intervals  $\beta_i$ , and the segment  $\gamma$  into the corresponding intervals  $\gamma_i$ , and consider the whole problem separately for each pair  $\beta_i\gamma_i$  of corresponding intervals. Again it will be necessary to take as  $B$  the rightmost point of the interval  $\beta_i$ , and as  $C$  the leftmost point of the interval  $\gamma_i$ , and then the minimum over all such pairs will give a lower estimate of the number  $S_I$ . Note

that our problem is further complicated by the fact that the estimate of the sum  $S_{\Gamma}$  also depends on the vector  $OA$ , and therefore everything just presented must

be carried out for various intervals of variation of  $\lambda$ , and not only for a fixed  $\lambda$ . This whole method can be described by partitioning the domain of reduction of forms into elementary subdomains. In this way, for  $m \geq 3$  one succeeds in obtaining the estimate  $\lambda_C \geq \sqrt[4]{3}/4$ .

**3°. On the constant  $\lambda_D$ .** Every equiaffine transformation of the plane that leaves the point  $O$  fixed is, in its normal form, a hyperbolic rotation with respect to certain mutually perpendicular asymptotes passing through the point  $O$ , plus, perhaps, an ordinary rotation about the point  $O$ , which no longer affects the sums  $S_m, g$ . We shall show that, using the constant  $\lambda_C$ , one can find such a constant  $\lambda_D$  that if  $\lambda < \lambda_D$  is the coefficient of contraction in this hyperbolic rotation, then already  $S_{2\Gamma'} > 7.714\dots$ . Of all affine transformations of the lattice  $\Gamma_1^2$  into a lattice  $\Gamma'$ , we shall consider the one which takes the reduced (in the sense of Lagrange) frame of the lattice  $\Gamma_1^2$ , composed of two sides of an equilateral triangle, into the reduced basic frame  $\mathfrak{A}_1, \mathfrak{A}_2$  of the lattice  $\Gamma'$ , where  $|\mathfrak{A}_1| = \lambda_C$ . In this case the circle circumscribed about the first regular triangle of the lattice  $\Gamma_1^2$  will pass into an ellipse with center at the point  $O$  and passing through the ends of the vectors  $\mathfrak{A}_1, \mathfrak{A}_2$ , and  $\mathfrak{A}_1 - \mathfrak{A}_2$ . It is easy to see that the minor semiaxis of this ellipse will be the smallest if the vectors  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are perpendicular. It is not hard to write down the equation of this ellipse, and then we find that if  $|\mathfrak{A}_1| = \lambda_C = 0.8579$ , then the minor semiaxis, just equal to the constant  $\lambda$ , is  $\lambda_D = 0.7502$ . Carrying out entirely analogous computations for  $m = 3, 4, \dots$ , we obtain respectively that  $\lambda_D > 1/\sqrt[4]{3}$ .

**4°. The hodograph of those hyperbolic rotations for which the term contributed by the equiaffine image of a regular triangle with center at the point  $O$  and side equal to one is equal to the term contributed by this triangle itself.**

We shall characterize every hyperbolic rotation about the point  $O$  by the vector issuing from the point  $O$ , going along that one of its asymptotes along which contraction occurs, and of length equal to the coefficient  $\lambda$  of this contraction. Let the origin lie at the point  $O$ , and let the asymptote along which contraction occurs make an angle  $\varphi$  with the  $x$ -axis; one of the vertices of the triangle lies on the negative ray of the  $y$ -axis, and the radius of the circle circumscribed about this triangle is equal to one. Then the coordinates (with respect to the asymptotes) of the vertices of the image of our triangle are

$$(\lambda \cos(30^\circ + \varphi); \frac{1}{\lambda} \sin(30^\circ + \varphi)); \quad (\lambda \cos(150^\circ + \varphi), \frac{1}{\lambda} \sin(150^\circ + \varphi)); \quad (\lambda \cos(270^\circ + \varphi), \frac{1}{\lambda} \sin(270^\circ + \varphi)),$$

and the sum of the reciprocals of the fourth powers of their distances from the point  $O$  is equal to

$$\frac{1}{(\lambda^2\alpha + \lambda^{-2}\bar{\alpha})^2} + \frac{1}{(\lambda^2\beta + \lambda^{-2}\bar{\beta})^2} + \frac{1}{(\lambda^2\gamma + \lambda^{-2}\bar{\gamma})^2},$$

Fig. 2

Figure 2: Fig. 2

where

$$\begin{aligned} \alpha &= \cos^2(30^\circ + \varphi); & \beta &= \cos^2(150^\circ + \varphi); & \gamma &= \cos^2(270^\circ + \varphi); \\ \bar{\alpha} &= \sin^2(30^\circ + \varphi); & \bar{\beta} &= \sin^2(150^\circ + \varphi); & \bar{\gamma} &= \sin^2(270^\circ + \varphi). \end{aligned}$$

If we put  $\lambda^4 = t$ ,  $t - 1 = \tau$ , and equate this sum to the analogous sum for the original triangle (equal to 3), then we obtain the equation of the hodograph

$$\frac{\tau + 1}{(\tau\alpha + 1)^2} + \frac{\tau + 1}{(\tau\beta + 1)^2} + \frac{\tau + 1}{(\tau\gamma + 1)^2} - 3 = 0.$$

Taking into account the identities  $\alpha + \beta + \gamma = 3/2$ ,  $\alpha\beta + \alpha\gamma + \beta\gamma = 9/16$ ,  $16\alpha\beta\gamma = \sin^2 3\varphi = \mu$ , after some computations we obtain an equation of the 6th degree in  $\tau$  with coefficients depending on  $\mu$ . This equation has no constant term and no term of the first degree with respect to ...

...namely,  $\tau$ , i.e.,  $\tau$  is its double root (corresponding to  $t = 1$ , i.e., to the original regular triangle itself). Hence the proof of Lemma 1 is obtained once again. Excluding this double root, we finally obtain the equation  $\tau^4\mu^2 + A\mu - B = 0$ .

Here  $A = 34\tau^3 + 96\tau^2 + 64\tau$  and  $B = 27\tau^3 + 90\tau^2 + 96\tau + 32$ . This is the equation of the desired hodograph in another form. Denote its left-hand side by  $\Phi$ . Differentiating  $\Phi$  with respect to  $\varphi$ , we obtain

$$\frac{\partial\Phi}{\partial\tau} \frac{d\tau}{d\varphi} + \frac{\partial\Phi}{\partial\mu} \frac{d\mu}{d\varphi} = 0$$

and find the extremum of  $\tau$  (i.e., of  $\lambda$ ) with respect to  $\varphi$ , putting  $d\tau/d\varphi = 0$ . Then either  $\partial\Phi/\partial\mu = 0$ , or  $d\mu/d\varphi = 0$ . But  $\partial\Phi/\partial\mu = 2\tau^4\mu + A = 0$  gives  $\mu = -A/2\tau^4$ , or, after substitution into the equation  $\tau^4\mu^2 + A\mu - B = 0$ , gives  $A^2 + 4\tau^4B = 0$ . All coefficients of this equation with respect to  $\tau$  are positive and, consequently, it has no positive roots; meanwhile, one may seek only an extremum for positive  $\tau$ , since to every compression along one asymptote there corresponds the inverse stretching along the other (negative  $\tau > -1$  correspond to  $0 < \lambda < 1$ , and positive ones to  $\lambda > 1$ ), and the hodograph of compressions is obtained from the hodograph of stretchings if the latter is inverted in the unit circle and rotated by  $90^\circ$ .

**Fig. 2**

Thus, the positive extrema of  $\tau$  are obtained only when  $d\mu/d\varphi = 0$ , i.e., when  $\varphi = k \cdot 30^\circ$  ( $k = 1, 3, 5, 7, 9, 11$ ). Thus the hodograph has the form shown in Fig. 2, and the radius of the circle circumscribed about its inner branch is

$$\lambda_0 = \frac{1}{\sqrt[4]{3}} = 0.7599 \dots$$

If  $m > 2$ , then it can be shown that the corresponding hodograph lies inside the hodograph for  $m = 2$ . If  $m \geq 3$ , then already  $\lambda_D > \lambda_0$ , and the theorem is therefore proved.

**5°. Application of the double triangle for  $m = 2$ .** Add to the first regular triangle of the lattice  $\Gamma_1^2$  a second one, next in size, concentric with it and rotated relative to it by  $30^\circ$ . The hodograph of this triangle is exactly the same as that of the first, but rotated by  $90^\circ$ , and the term contributed by it has weight  $1/9$  relative to the first. Studying in detail what the first triangle contributes to the shaded crescent and what the second contributes, we convince ourselves, with the aid of rather rough estimates, that the quantity which the first triangle fails to contribute (in comparison with the regular one) is covered by the quantity which the second contributes (in comparison with the regular one). This completes the proof of the theorem for  $m = 2$ , since the entire lattice  $\Gamma_1^2$  can be decomposed into such pairs of triangles having no common points pairwise.

**Remark.** After the proof of the theorem it follows that the numbers  $\lambda_C$  and  $\lambda_D$ , for any  $m \geq 2$ , are in fact equal to one.

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*Note: Figure translations are in progress. See original paper for figures.*

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