P. I. Lizorkin

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Abstract

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P. I. Lizorkin

ON THE FOURIER TRANSFORM IN BESOV SPACES. THE ZERO SCALE $B^0_{p,\theta}$

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1. The spaces $B^{(r_1,\dots,r_n)}_{p,\theta}$, $r_j > 0$, $j = 1,\dots,n$, $1 \le p,\theta \le \infty$, introduced in full generality by O. V. Besov ([1]) (see also the survey ([2])), are important for analysis and its applications. Recently they have served as the subject of further investigations ([3–5]). In papers ([3,5]), in particular, the spaces $B^{(r,\dots,r)}_{p,\theta} = B^r_{p,\theta}$ were defined for $r \le 0$, and various of their characteristics were given. There an isomorphism was also established between the spaces $B^r_{p,\theta}$ with different r. In the present note the spaces $B^{(r_1,\dots,r_n)}_{p,\theta}$, $-\infty < r_j < \infty$, $1 < p < \infty$, $1 \le \theta < \infty$, are considered from the point of view of the Fourier transform.

2. Denote by R_n the n-dimensional space of points $x = (x_1,\dots,x_n)$, and let $r_j > 0$, $r_j = \bar r_j + \alpha_j$, where $\bar r_j$ is an integer, $0 < \alpha_j \le 1$. Recall that a function $f(x)$, by definition, belongs to the space $B^{(r)}_{p,\theta} = B^{(r_1,\dots,r_n)}_{p,\theta}$, $1 \le p \le \infty$, $1 \le \theta < \infty$, if its norm is finite,

$$|f, B^{(r)}_{p,\theta}| = \|f\|_{L_p(R_n)} + \sum_{j=1}^n I_{r_j;p,\theta},$$

$$I^\theta_{r_j;p,\theta} = \int_0^\infty \left\| \frac{\partial^{\bar r_j} f}{\partial x_j^{\bar r_j}}(x_1,\dots,x_j+t,\dots,x_n) - 2\frac{\partial^{\bar r_j} f}{\partial x_j^{\bar r_j}}(x_1,\dots,x_n) + \frac{\partial^{\bar r_j} f}{\partial x_j^{\bar r_j}}(x_1,\dots,x_j-t,\dots,x_n)\right\|^\theta_{L_p} \frac{dt}{t^{1+\theta\alpha_j}}.$$

In the case $\alpha_j < 1$, instead of the second difference one may use the first:

$$\frac{\partial^{\bar r_j} f}{\partial x_j^{\bar r_j}}(x_1,\dots,x_j+t,\dots,x_n) - \frac{\partial^{\bar r_j} f}{\partial x_j^{\bar r_j}}(x_1,\dots,x_n).$$

Fix a number $b > 1$, and let $a_j^{r_j} = b$. Introduce the notation: $\lambda = (\lambda_1,\dots,\lambda_n)$ is the independent variable in the space of Fourier transforms; $\tilde f(\lambda)$ is the Fourier transform of the function $f(x)$; $D_{a^k} = D_{a_1^k,\dots,a_n^k}$ is the parallelepiped $\{-a_j^k \le \lambda_j \le a_j^k,\ j = 1,\dots,n\}$. Put

$$S_{a^0}(x) = \frac{1}{(2\pi)^{n/2}} \int_{D_{a^0}} \tilde{f}(\lambda) e^{i\lambda x}\, d\lambda, \qquad f_{a^k}(x) = \frac{1}{(2\pi)^{n/2}} \int_{D_{a^k} - D_{a^{k-1}}} \tilde{f}(\lambda) e^{i\lambda x}\, d\lambda.$$

We proceed from the following assertions:

Theorem 1*. The function $f(x)$ belongs to the space $B^{(r)}_{p,\theta}$, $r_j > 0$, $1 < p < \infty$, $1 \le \theta < \infty$, if and only if the series converges

$$\|S_{a^0}\|^\theta_{L_p} + \sum_{k=1}^{\infty} b^{k\theta} \|f_{a^k}\|^\theta_{L_p}. \tag{1}$$

* A similar characterization of the spaces $B^{(r)}_{p,\theta}$ in terms of approximations constructed with the aid of Jackson kernels was first given explicitly by T. I. Amanov ([6]); it can also be stated in terms of best approximations.

Here there exist constants c_1, c_2, independent of f, such that

$$c_1 |f, B^{(r)}_{p,\theta}| \le \left\{ \|S_{a^0}\|^\theta_{L_p} + \sum_{k=1}^{\infty} b^{k\theta} \|f_{a^k}\|^\theta_{L_p} \right\}^{1/\theta} \le c_2 |f, B^{(r)}_{p,\theta}|. \tag{2}$$

Theorem 2. Let $f \in L_p(R_n)$. Then there exist constants c_1, c_2, independent of f, such that

$$c_1 \sum_{j=1}^{n} I^\theta_{r_j;p,\theta} \le \sum_{k=-\infty}^{\infty} b^{k\theta} \|f_{a^k}\|^\theta_{L_p} \le c_2 \sum_{j=1}^{n} I^\theta_{r_j;p,\theta}. \tag{3}$$

3. Let us first consider the case where $r_1 = r_2 = \cdots = r_n = r > 0$. Take $a = 2$, $b = 2^r$, and, simplifying the notation, set

$$D_{a^0} = \Gamma_0, \quad S_{a^0} = f_0; \quad D_{a^k} - D_{a^{k-1}} = \Gamma_k, \quad f_{a^k} = f_k, \quad k = 1, 2, \dots.$$

On the basis of Theorem 1 we can introduce in $B^{(r,\dots,r)}_{p,\theta} = B^r_{p,\theta}$ the equivalent norm

$$\{f, B^r_{p,\theta}\} = \left\{ \sum_{k=0}^{\infty} 2^{kr\theta} \|f_k\|^\theta_{L_p} \right\}^{1/\theta}. \tag{4}$$

Expression (4) can also be given meaning for $r \le 0$. Let S and S' be the spaces of basic and generalized functions (S'-distributions) of L. Schwartz.

Definition 1. An S'-distribution $f(x)$ belongs to the space $B^r_{p,\theta}$, $-\infty < r < \infty$, $1 < p < \infty$, $1 \le \theta < \infty$, if it is representable in the form

$$f(x) = \sum_{k=0}^{\infty} f_k(x), \tag{5}$$

where $f_k(x)$ are entire functions of degree $\leq 2^k$ in each of the variables, whose Fourier transform is concentrated in Γ_k, satisfying the estimate

$$\{f, B^r_{p,\theta}\} = \left\{ \sum_{k=0}^{\infty} 2^{kr\theta} \|f_k(x)\|^{\theta}_{L_p} \right\}^{1/\theta} < \infty. \tag{6}$$

For $r > 0$, this definition, with the norm introduced by relation (6), coincides with the original one, and therefore we keep the notation. Below we shall see that the spaces $B^r_{p,\theta}$ with norm (6) are Banach spaces also for $r \leq 0$; up to equivalence of norms, they coincide with the spaces B^r introduced in ([5]) and ([3]) (where they are denoted by $\Lambda(r; p, \theta)$).

We shall call the space $B^0_{p,\theta}$ the **zero** (p, θ)**-space**. The role of this space is comparable with the role of the Lebesgue space L_p (we shall clarify this statement later). We note that zero spaces close to $B^0_{p,\theta}$ were recently studied by M. Ramazanov ([7]).

Theorem 3. Let p, θ be fixed. The spaces $B^r_{p,\theta}$ with different r are isomorphic to one another. The isomorphism of the spaces $B^0_{p,\theta}$, $B^r_{p,\theta}$ is effected by the operator defined in Fourier images by the equality

$$\tilde{g}(\lambda) = \left(1 + |\lambda|^2\right)^{r/2} \tilde{f}(\lambda), \quad f \in B^r_{p,\theta}, \quad g \in B^0_{p,\theta}. \tag{7}$$

This theorem makes it possible to assert the coincidence of the space $B^0_{p,\theta}$ with the zero space introduced in ([3,5]), and gives an independent proof of the isomorphism theorem established there. From the indicated isomorphism follows the completeness of the space for all real r (since completeness for positive r was established in (1)). Also valid is the assertion of density in $B^r_{p,\theta}$, for $1 \leq \theta < \infty$, of the set C_0^{∞} of infinitely differentiable functions with compact support, of the set L_p of entire functions of finite degree belonging to $L_p(R_n)$, and of the set of functions from S. It follows from the following facts:
1) under an isomorphism, an everywhere dense set preserves the property

density; 2) mappings of the form (8) carry $_p$ and S into themselves; 3) $_p$ and S are dense in $B^r_{p,\theta}$, $r > 0$, $1 \leq \theta < \infty$ ([1,3]).

Remark 1. The density of S in $B^r_{p,\theta}$ makes it possible effectively to find the decomposition (5) for a given function f. The point is that the construction of $\tilde{f}_k(x)$ is formally reduced to multiplying $\tilde{f}$ by the characteristic function χ_k of the domain Γ_k, but within the framework of S'-distributions such multiplication is not defined. However, on functions $\varphi \in S$, multiplication of $\tilde{\varphi}$ by χ_k gives rise to an operator bounded in the metric of $B^r_{p,\theta}$. The closure of this operator leads to a correct definition of $\tilde{f}_k$ from f.

Let us also note that between the space $(B^r_{p,\theta})^*$, conjugate to $B_{p,\theta}$, and the space $B^{-r}_{p',\theta'}$, $1/p + 1/p' = 1/\theta + 1/\theta' = 1$, there exists an isomorphism established by the formula

$$\langle f, g\rangle = \sum_{k=0}^{\infty} \int_{R_n} f_k(x) g_k(x)\, dx.$$

4. We pass to the general case.

Definition 2. An S'-distribution f belongs to the space $\mathfrak{B}^{(\mathbf{r})}_{p,\theta}$, $-\infty < r_j < \infty$, $j = 1, \dots, n$, $1 < p < \infty$, $1 \le \theta < \infty$, if there exists an element $h \in B^0_{p,\theta}$ such that

$$\tilde{h}(\lambda) = \sum_{j=1}^{n} (1 + \lambda_j^2)^{r_j/2} \tilde{f}(\lambda).$$

For positive r_j, the space $\mathfrak{B}^{(\mathbf{r})}_{p,\theta}$ coincides with $B^{(\mathbf{r})}_{p,\theta}$ up to equivalence of norms, if one sets $\{f, \mathfrak{B}^{(\mathbf{r})}_{p,\theta}\} = \{h, B^0_{p,\theta}\}$. The space $\mathfrak{B}^{(r,\dots,r)}_{p,\theta}$ for $r < 0$ is equivalent to $B^r_{p,\theta}$. These assertions, as well as Theorem 3, are proved with the aid of Theorem 5 on multipliers (see below) and estimates of trigonometric integrals of Bernstein-inequality type.

All the spaces $\mathfrak{B}^{(\mathbf{r})}_{p,\theta}$ with different $\mathbf{r}$ are isomorphic to one another by virtue of the definition. Hence it follows that they are Banach spaces for arbitrary r_j. The assertion on the density in $\mathfrak{B}^{(\mathbf{r})}_{p,\theta}$ of the spaces C_0^∞, ${}_p$, and S is also preserved; $(\mathfrak{B}^{(\mathbf{r})}_{p,\theta})^* = \mathfrak{B}^{(-\mathbf{r})}$. In the spaces $\mathfrak{B}^{(\mathbf{r})}_{p,\theta}$, for fixed $\mathbf{r}$ and θ, interpolation theorems of Riesz and Marcinkiewicz type, known for the scale L_p, hold. A function Φ that is a multiplier of type (L_p, L_q) (see Theorem 5 and (8)) represents a multiplier of type $(\mathfrak{B}^{(\mathbf{r})}_{p,\theta}, \mathfrak{B}^{(\mathbf{r})}_{q,\theta})$.

5. Let us compare the spaces $B^{(\mathbf{r})}_{p,\theta}$ with the spaces L^r_p (9), consisting of functions $f \in L_p$ whose generalized Liouville (with respect to the individual variables) derivatives $\partial^{r_j} f/\partial x_j^{r_j}$, $j = 1, \dots, n$, belong to L_p. Recall that the Liouville derivative $\partial^s f/\partial x_j^s$ of order s of a function f with respect to the variable x_j is formally defined by the equality $\partial^s f/\partial x^s = (i\lambda_j)^s \tilde{f}(\lambda)$ (see (9)); for integer s this is the usual derivative. Recall also that for integer $r_1 = \dots = r_n = l$ the space $L^{(l,\dots,l)}_p$ coincides with the Sobolev space $W^{(l)}_p$ (10).

Theorem 4. *A function $f \in B^{(\mathbf{r})}_{p,\theta}$, $r_j > 0$, if and only if $f \in B^0_{p,\theta}$; $\partial^{r_j} f/\partial x_j^{r_j} \in B^0_{p,\theta}$, $j = 1, \dots, n$. Moreover,*

$$\{f, B^{(\mathbf{r})}_{p,\theta}\} \sim \{f, B_{p,\theta}\} + \sum_{j=1}^{n} \left\{ \frac{\partial^{r_j} f}{\partial x_j^{r_j}}, B^0_{p,\theta} \right\}.$$

This proposition reveals the complete analogy in the construction of the spaces L_p^r and $B^{(\mathbf{r})}$: over the "zero" scale (L_p, B) a "differentiable superstructure" is erected. It is known ([1],[9]) that, with respect to boundary embedding theorems, the spaces $L_p^{(\mathbf{r})}$ and $B^{(\mathbf{r})}_{p,\theta}$ behave differently. The question arises: what properties of the zero scale ensure the closedness (in the sense of S. M. Nikol' skii (11)) of the system of embedding theorems?

Finally, let us note the relations between $L_p^{(r)}$ and $B^{(r)}_{p,\theta}$:

$$L_p \subset B^{(r)}_{p,\theta} \quad \text{for } \max(p,2) \le \theta < \infty,$$

$$B^{(r)}_{p,\theta} \subset L_p^{(r)} \quad \text{for } 1 \le \theta \le \min(p,2).$$

These embeddings are not continuous; they are strict if p and θ are different from 2; $L_2^{(r)} = B^{(r)}_{2,2}$.

6. It is useful to emphasize that the characterization of the spaces $B^{(r)}_{p,\theta}$ in terms of the Fourier transform opens up new possibilities for their use in applications (until now, the spaces used predominantly were

$$B^r_{p,p} = B^r_p, \qquad \text{and also} \qquad B^{(r_1,\dots,r_n)}_{2,2}).$$

The equivalence relations between the different norms of f (see (1–5)) and the quantity $\{f, \mathfrak{B}^{(r)}_{p,\theta}\}$ may be interpreted as generalizations of Plancherel's theorem to Besov spaces. For example, on the basis of Theorem 2 we obtain, for $n = 1$, $0 < r \le 1$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left|f(x) - 2f\left(\frac{1}{2}(x+y)\right) + f(y)\right|^p}{|x-y|^{1+pr}}\,dx\,dy \sim \sum_{k=-\infty}^{\infty} \left\| \left(\frac{d^r f}{dx^r}\right)_{2^k} \right\|_{L_p}^p .$$

For $p = 2$ this equivalence relation turns into the equality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left|f(x) - 2f\left(\frac{1}{2}(x+y)\right) + f(y)\right|^2}{|x-y|^{1+2r}}\,dx\,dy = C(r)\int_{-\infty}^{\infty} |\lambda|^{2r} |\tilde{f}(\lambda)|^2\,d\lambda.$$

7. In conclusion we formulate a theorem on multipliers, which is also of independent interest.

Theorem 5. *Let the function $\Phi(\lambda)$ in any parallelepiped*

$$U_m\{\lambda;\ 2^{m_k} < |\lambda_k| \le 2^{m_k+1},\ k = 1, \dots, n\}, \qquad m_k = 0, \pm 1, \pm 2, \dots,$$

be representable in the form

$$\Phi(\lambda) = \int_{-\infty}^{\lambda_1} \cdots \int_{-\infty}^{\lambda_n} d\mu_m, \qquad \lambda \in U_m, \tag{8}$$

where μ_m are finite measures for which

$$\sup_m \operatorname{var} \mu_m = \sup_m \int_{R_n} |d\mu_m| \le M.$$

Then the transformation $T : f \to g$, defined by the equality $Tf = \Phi(\lambda)\tilde{f} = g$, is a bounded transformation from $L_p(R_n)$ into $L_p(R_n)$, $1 < p < \infty$.

A theorem on multipliers from $L_p(R_n)$ into $L_q(R_n)$, $p \le q$, is formulated analogously, with the representation (8) replaced by

$$\Phi(\lambda) = \int_{-\infty}^{\lambda_1} \cdots \int_{-\infty}^{\lambda_n} \frac{d\mu_m(t_1, \dots, t_n)}{(\lambda_1 - t_1)^\beta \cdots (\lambda_n - t_n)^\beta}, \qquad \beta = \frac{1}{p} - \frac{1}{q}, \quad \lambda \in U_m.$$

The indicated lacunary decomposition of R_n into parallelepipeds allows shifts and deformations.

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Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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References

1. O. V. Besov, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, **60**, 42 (1961).

2. S. M. Nikol' skii, UMN, **16**, 5, 63 (1961).

3. M. N. Taibleson, J. Math. and Mech., **13**, 407 (1964).

4. N. Aronszajn, F. Mulla, P. Szeptycki, Ann. Inst. Fourier, **13**, 2, 211 (1963).

5. S. M. Nikolski, I. L. Lions, P. I. Lizorkin, Ann. Sc. Norm. Sup. Pisa, ser. III (**1965**).

6. T. I. Amanov, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, **77**, 5 (1965).

7. M. Ramazanov, DAN, **161**, No. 3 (1965).

8. P. I. Lizorkin, DAN, **152**, No. 4, 808 (1963).

9. P. I. Lizorkin, Matem. sborn., **60** (102), 325 (1963).

10. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

11. S. M. Nikol' skii, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, **38**, 244 (1951).

Note: Figure translations are in progress. See original paper for figures.